INTRODUCTION TO PROBABILITY

by

Dimitri P. Bertsekas and John N. Tsitsiklis

CHAPTER 3: ADDITIONAL PROBLEMS

SECTION 3.1. Continuous Random Variables and PDFs

Problem 1. The runner-up in a road race is given a reward that depends on the difference between his time and the winner's time. He is given 10 dollars for being one minute behind, 6 dollars for being one to three minutes behind, 2 dollars for being 3 to 6 minutes behind, and nothing otherwise. Given that the difference between his time and the winner's time is uniformly distributed between 0 and 12 minutes, find the mean and variance of the reward of the runner-up.

Problem 2. Let X be a random variable with PDF

$$f_X(x) = \begin{cases} 2x/3 & \text{if } 1 < x \leq 2, \\ 0 & \text{otherwise,} \end{cases}$$

and let $Y = X^2$. Calculate $\mathbf{E}[Y]$ and $\text{var}(Y)$.

SECTION 3.2. Cumulative Distribution Functions

Problem 3. Find the PDF, the mean, and the variance of the random variable X with CDF

$$F_X(x) = \begin{cases} 1 - \dfrac{a^3}{x^3}, & \text{if } x \geq a, \\ 0, & \text{if } x < a, \end{cases}$$

where a is a positive constant.

Problem 4. You are allowed to take a certain test three times, and your final score will be the maximum of the test scores. Your score in test i, where $i = 1, 2, 3$, takes one of the values from i to 10 with equal probability $1/(11 - i)$, independently of the scores in other tests. What is the PMF of the final score?

Problem 5. The **median** of a random variable X is a number μ that satisfies $F_X(\mu) = 1/2$. Find the median of the exponential random variable with parameter λ.

SECTION 3.3. Normal Random Variables

Problem 6. A radar tends to overestimate the distance of an aircraft, and the error is a normal random variable with a mean of 50 meters and a standard deviation 100 meters. What is the probability that the measured distance will be smaller than the true distance?

Problem 7. Let X be normal with mean 1 and variance 4. Let $Y = 2X + 3$.

(a) Calculate the PDF of Y.

(b) Find $\mathbf{P}(Y \geq 0)$.

Problem 8. A signal of amplitude $s = 2$ is transmitted from a satellite but is corrupted by noise, and the received signal is $Z = s + W$, where W is noise. When the weather is good, W is normal with zero mean and variance 1. When the weather is bad, W is normal with zero mean and variance 4. Good and bad weather are equally likely. In the absence of any weather information:

(a) Calculate the PDF of X.

(b) Calculate the probability that X is between 1 and 3.

Problem 9. Oscar uses his high-speed modem to connect to the internet. The modem transmits zeros and ones by sending signals -1 and $+1$, respectively. We assume that any given bit has probability p of being a zero. The telephone line introduces additive zero-mean Gaussian (normal) noise with variance σ^2 (so, the receiver at the other end receives a signal which is the sum of the transmitted signal and the channel noise). The value of the noise is assumed to be independent of the encoded signal value.

(a) Let a be a constant between -1 and 1. The receiver at the other end decides that the signal -1 (respectively, $+1$) was transmitted if the value it receives is less (respectively, more) than a. Find a formula for the probability of making an error.

(b) Find a numerical answer for the question of part (a) assuming that $p = 2/5$, $a = 1/2$ and $\sigma^2 = 1/4$.

SECTION 3.4. Conditioning on an Event

Problem 10. An old modem can take anywhere from 0 to 30 seconds to establish a connection, with all times between 0 and 30 being equally likely.

(a) What is the probability that if you use this modem you will have to wait more than 15 seconds to connect?

(b) Given that you have already waited 10 seconds, what is the probability of having to wait at least 10 more seconds?

Problem 11. Consider a random variable X with PDF

$$f_X(x) = \begin{cases} 2x/3, & \text{if } 1 < x \leq 2, \\ 0, & \text{otherwise,} \end{cases}$$

and let A be the event $\{X \geq 1.5\}$. Calculate $\mathbf{E}[X]$, $\mathbf{P}(A)$, and $\mathbf{E}[X \mid A]$.

Problem 12. Dino, the cook, has good days and bad days with equal frequency. On a good day, the time (in hours) it takes Dino to cook a souffle is described by the PDF

$$f_G(g) = \begin{cases} 2, & \text{if } 1/2 < g \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

but on a bad day, the time it takes is described by the PDF

$$f_B(b) = \begin{cases} 1, & \text{if } 1/2 < b \le 3/2, \\ 0, & \text{otherwise.} \end{cases}$$

Find the conditional probability that today was a bad day, given that it took Dino less than three quarters of an hour to cook a souffle?

Problem 13. One of two wheels of fortune, A and B, is selected by the toss of a fair coin, and the wheel chosen is spun once to determine the value of a random variable X. If wheel A is selected, the PDF of X is

$$f_{X|A}(x|A) = \begin{cases} 1 & \text{if } 0 < x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

If wheel B is selected, the PDF of X is

$$f_{X|B}(x|B) = \begin{cases} 3 & \text{if } 0 < w \le 1/3, \\ 0 & \text{otherwise.} \end{cases}$$

If we are told that the value of X was less than $1/4$, what is the conditional probability that wheel A was the one selected?

SECTION 3.5. Multiple Continuous Random Variables

Problem 14. Alexei is vacationing in Monte Carlo. The amount X (in dollars) he takes to the casino each evening is a random variable with a PDF of the form

$$f_X(x) = \begin{cases} ax, & \text{if } 0 \le x \le 40, \\ 0, & \text{otherwise.} \end{cases}$$

At the end of each night, the amount Y that he has when leaving the casino is uniformly distributed between zero and twice the amount that he came with.

(a) Determine the joint PDF $f_{X,Y}(x, y)$.

(b) What is the probability that on a given night Alexei makes a positive profit at the casino?

(c) Find the PDF of Alexei's profit $Y - X$ on a particular night, and also determine its expected value.

Problem 15. A family has three children, A, B, and C, of height X_1, X_2, X_3, respectively. If X_1, X_2, X_3 are independent and identically distributed continuous random variables, evaluate the following probabilities:

(a) $\mathbf{P}(A \text{ is the tallest child})$.

(b) $\mathbf{P}(A \text{ is taller than } B \mid A \text{ is taller than } C)$.

(c) $\mathbf{P}(A \text{ is taller than } B \mid B \text{ is taller than } C)$.

(d) $\mathbf{P}(A \text{ is taller than } B \mid A \text{ is shorter than } C)$.

(e) $\mathbf{P}(A \text{ is taller than } B \mid B \text{ is shorter than } C)$.

Problem 16. Let X have a uniform distribution in the unit interval $[0, 1]$, and let Y have an exponential distribution with parameter $\nu = 2$. Assume that X and Y are independent. Let $Z = X + Y$.

(a) Find $\mathbf{P}(Y \geq X)$.

(b) Find the conditional PDF of Z given that $Y = y$.

(c) Find the conditional PDF of Y given that $Z = 3$.

Problem 17. Let X and Y be independent random variables, with each one uniformly distributed in the interval $[0, 1]$. Find the probability of each of the following events.

(a) $X > 6/10$.

(b) $Y < X$.

(c) $X + Y \leq 3/10$.

(d) $\max\{X, Y\} \geq 1/3$.

(e) $XY \leq 1/4$.

Problem 18. Let P a random variable which is uniformly distributed between 0 and 1. On any given day, a particular machine is functional with probability P. Furthermore, given the value of P, the status of the machine on different days is independent

(a) Find the probability that the machine is functional on a particular day.

(b) We are told that the machine was functional on m out of the last n days. Find the conditional PDF of P. You may use the identity

$$\int_0^1 p^k (1-p)^{n-k} dp = \frac{k!(n-k)!}{(n+1)!}.$$

(c) Find the conditional probability that the machine is functional today given that it was functional on m out of the last n days.

SECTION 3.6. Derived Distributions

Problem 19. Let X be a random variable with PDF f_X. Find the PDF of the random variable $|X|$ in the following three cases.

(a) X is exponentially distributed with parameter λ.

(b) X is uniformly distributed in the interval $[-1, 2]$.

(c) f_X is a general PDF.

Problem 20. Your driving time to work is between 30 and 45 minutes if the day is sunny, and between 40 and 60 minutes if the day is rainy, with all times being equally likely in each case. Assume that a day is sunny with probability 2/3 and rainy with probability 1/3.

(a) Find the PDF, the mean, and the variance of your driving time.

(b) On a given day your driving time was 45 minutes. What is the probability that this particular day was rainy?

(c) Your distance to work is 20 miles. What is the PDF, the mean, and the variance of your average speed (driving distance over driving time)?

Problem 21. The random variables X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2, & \text{if } x > 0 \text{ and } y > 0 \text{ and } x + y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let A be the event $\{Y \leq 0.5\}$ and let B be the event $\{Y > X\}$.

(a) Calculate $\mathbf{P}(B \mid A)$.

(b) Calculate $f_{X|Y}(x \mid 0.5)$. Calculate also the conditional expectation and the conditional variance of X, given that $Y = 0.5$.

(c) Calculate $f_{X|B}(x)$.

(d) Calculate $\mathbf{E}[XY]$.

(e) Calculate the PDF of Y/X.