
INTRODUCTION TO PROBABILITY

by

Dimitri P. Bertsekas and John N. Tsitsiklis

CHAPTER 5: ADDITIONAL PROBLEMS[†]

Last updated: November 18, 2002

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SECTION 7.1. Some Useful Inequalities

Problem 1. Let X be a random variable and let α be a positive constant. Show that

$$\mathbf{P}(|X| \geq c) \leq \frac{\mathbf{E}[|X|^\alpha]}{c^\alpha}, \quad \text{for all } c > 0.$$

Solution. We have

$$\mathbf{P}(|X| \geq c) = \mathbf{P}(|X|^\alpha \geq c^\alpha) \leq \frac{\mathbf{E}[|X|^\alpha]}{c^\alpha},$$

where the last inequality is the Markov inequality applied to the random variable $|X|^\alpha$.

Problem 2.

- (a) Given the information $\mathbf{E}[X] = 7$ and $\text{var}(X) = 9$, use the Chebyshev inequality to find a lower bound for the probability $\mathbf{P}(4 < X < 10)$.
- (b) Find the smallest and the largest possible values of the probability $\mathbf{P}(4 < X < 10)$, given the mean and variance information from part (a).

Solution. (a) The Chebyshev inequality yields $\mathbf{P}(|X - 7| \geq 3) \leq 9/3^2 = 1$, which implies the uninformative bound $\mathbf{P}(4 < X < 10) \geq 0$.

(b) We will show that $\mathbf{P}(4 < X < 10)$ can be as small as 0, and can be made arbitrarily close to 1.

Consider a random variable that equals 4 with probability $1/2$, and 10 with probability $1/2$. This random variable has mean 7 and variance 9, and $\mathbf{P}(4 < X < 10) = 0$. Therefore, the lower bound from part (a) is the best possible.

Let us now fix a small positive number ϵ and another positive number c , and consider a discrete random variable X with PMF

$$p_X(x) = \begin{cases} 0.5 - \epsilon, & \text{if } x = 4 + \epsilon, \\ 0.5 - \epsilon, & \text{if } x = 10 - \epsilon, \\ \epsilon, & \text{if } x = 7 - c, \\ \epsilon, & \text{if } x = 7 + c. \end{cases}$$

This random variable has a mean of 7 (by symmetry). Its variance is

$$(0.5 - \epsilon)(3 - \epsilon)^2 + (0.5 - \epsilon)(3 + \epsilon)^2 + 2\epsilon c^2,$$

and can be made equal to 9 by suitably choosing c . For this random variable, we have $\mathbf{P}(4 < X < 10) = 1 - 2\epsilon$, which can be made arbitrarily close to 1.

On the other hand, $\mathbf{P}(4 < X < 10)$ cannot be made equal to 1. Indeed, if this probability was equal to 1, then we would have $|X - 7| < 3$, which would imply that the variance is less than 9.

Problem 3. Investigate whether the Chebyshev inequality is tight. That is, for every μ , σ , and $c \geq \sigma$, does there exist a random variable X with mean μ and standard deviation σ such that

$$\mathbf{P}(|X - \mu| \geq c) = \frac{\sigma^2}{c^2}?$$

Solution. Consider a random variable X with PMF

$$p_X(x) = \begin{cases} p, & \text{if } x = \mu - c, \\ p, & \text{if } x = \mu + c, \\ 1 - 2p, & \text{if } x = \mu. \end{cases}$$

The mean of X is μ . Its variance is $2pc^2$. This can be made equal to the given variance σ^2 by setting $p = \sigma^2/(2c^2)$. For this random variable, we have

$$\mathbf{P}(|X - \mu| \geq c) = 2p = \frac{\sigma^2}{c^2},$$

and therefore the Chebyshev inequality is tight.

Problem 4. Chernoff bound for a Poisson random variable. Let X be a Poisson random variable with parameter λ .

(a) Show that for every $s \geq 0$, we have

$$\mathbf{P}(X \geq k) \leq e^{\lambda(e^s - 1)} e^{-sk}.$$

(b) Assuming that $k > \lambda$, show that

$$\mathbf{P}(X \geq k) \leq \frac{e^{-\lambda} (e\lambda)^k}{k^k}.$$

Solution. (a) As shown in Problem 1 in Chapter 7 of the text, or by a simple application of the Markov inequality, we have

$$\mathbf{P}(X \geq k) \leq e^{-sk} \mathbf{E}[e^{sX}] = e^{\lambda(e^s - 1)} e^{-sk},$$

where the last equality made use of the formula for the transform associated with a Poisson random variable.

(b) The inequality of part (a) is true for every $s \geq 0$. To find the tightest inequality, we optimize with respect to s . By taking the derivative of the exponent, we find that the derivative is zero when $\lambda e^s = k$. Let us therefore use the bound of part (a) with the particular choice of s that satisfies $\lambda e^s = k$. Note that this value of s is positive under the assumption that $k > \lambda$. Using the substitutions $\lambda e^s = k$, $e^{-s} = \lambda/k$, we obtain the desired bound.

SECTION 7.2. The Weak Law of Large Numbers

Problem 5. Bo assumes that X , the height in meters of any Canadian selected by an equally likely choice among all Canadians, is a random variable with $\mathbf{E}[X] = h$. Because Bo is sure that no Canadian is taller than 3 meters, he decides to use 1.5 meters as a conservative value for the standard deviation of X . To estimate h , Bo uses the average of the heights of n Canadians he selects at random.

- In terms of h and Bo's 1.5 meter bound for the standard deviation of X , determine the expectation and standard deviation of H .
- Find as small a value of n as possible such that the standard deviation of Bo's estimator is guaranteed to be less than 0.01 meters.
- Bo would like to be 99% sure that his estimate is within 5 centimeters of the true average height of Canadians. Using the Chebyshev inequality, calculate the minimum value of n that will achieve this objective.
- If we agree that no Canadians are taller than three meters, why is it correct to use 1.5 meters as an upper bound on the standard deviation for X , the height of any Canadian selected at random?

Solution. (a) Let X_i be the height of the i th selected Canadian, and let H be the estimated height. We have $H = (X_1 + X_2 + \cdots + X_n)/n$, which yields

$$\mathbf{E}[H] = \frac{n\mathbf{E}[X]}{n} = h,$$

$$\sigma_H = \sqrt{\text{var}(H)} = \sqrt{\frac{n\text{var}(X)}{n^2}} = \frac{1.5}{\sqrt{n}}.$$

(b) The stated condition translates to $1.5/\sqrt{n} < 0.01$. We solve for n and obtain $n > 22,500$.

(c) We have

$$\mathbf{P}(|H - \mathbf{E}[H]| \geq t) \leq \left(\frac{\sigma_H}{t}\right)^2,$$

or

$$\mathbf{P}(|H - \mathbf{E}[H]| \leq t) \geq 1 - \left(\frac{\sigma_H}{t}\right)^2.$$

We use the values $t = 0.05$, $\sigma_H = 1.5/\sqrt{n}$, together with the requirement

$$1 - \left(\frac{\sigma_H}{t}\right)^2 \geq 0.99,$$

to obtain

$$n \geq \left(\frac{1.5}{0.05}\right)^2 \frac{1}{0.01} = 90,000.$$

(d) The random variable X has an unknown distribution over the interval $[0, 3]$. The variance of a random variable increases as its distribution becomes more spread out. Thus, the maximal variance will be obtained if the random variable X can only take the values 0 and 3. With such a distribution, and with $p = \mathbf{P}(X = 3)$, the variance becomes $\mathbf{E}[X^2] - (\mathbf{E}[X])^2 = 9p - 9p^2$. This expression is largest when $p = 1/2$, yielding a variance of $9/4$ and a standard deviation of $3/2$.

SECTION 7.3. Convergence in Probability

Problem 6. Let $X_1, X_2, \dots$ be independent, identically distributed random variables with $\mathbf{E}[X] = 2$ and $\text{var}(X) = 9$, and let $Y_i = X_i/2^i$. We also define T_n and A_n to be the sum and the sample mean, respectively, of the random variables $Y_1, \dots, Y_n$.

- (a) Evaluate the mean and variance of Y_n , T_n , and A_n .
- (b) Does Y_n converge in probability? If so, to what value?
- (c) Does T_n converge in probability? If so, to what value?
- (d) Does A_n converge in probability? If so, to what value?

Solution. (a) We have

$$\mathbf{E}[Y_n] = \mathbf{E}[X_n] \cdot 2^{-n} = 2^{1-n}, \quad \text{var}(Y_n) = \text{var}(X_n) \cdot 9 \cdot 2^{-2n} = 9 \cdot 4^{-n},$$

$$\mathbf{E}[T_n] = \mathbf{E}[Y_1 + \dots + Y_n] = 2 \sum_{i=1}^n 0.5^i = 2 \frac{0.5(1 - 0.5^n)}{1 - 0.5} = 2 - 0.5^{n-1},$$

$$\text{var}(T_n) = \text{var}(Y_1 + \dots + Y_n) = 9 \sum_{i=1}^n 4^{-i} = 9 \left(\frac{\frac{1}{4}(1 - (\frac{1}{4})^n)}{1 - \frac{1}{4}} \right) = 3 \left(1 - \left(\frac{1}{4} \right)^n \right),$$

$$\mathbf{E}[A_n] = \mathbf{E} \left[\frac{1}{n} T_n \right] = \frac{1}{n} \mathbf{E}[T_n] = \frac{2}{n} \left(1 - \left(\frac{1}{2} \right)^n \right),$$

$$\text{var}(A_n) = \text{var} \left(\frac{1}{n} T_n \right) = \left(\frac{1}{n} \right)^2 \text{var}(T_n) = \frac{3}{n^2} \left(1 - \left(\frac{1}{4} \right)^n \right).$$

- (b) Yes. As n increases, the expected value of Y_n approaches 0, and the variance of Y_n approaches 0. By the Chebyshev inequality, Y_n converges to 0, in probability.
- (c) No. As n increases, more independent terms are being added, which increase the variance of T_n and spread out its distribution.
- (d) Yes. As n increases, the expected value of A_n approaches 0, and the variance of A_n approaches 0. By the Chebyshev inequality, A_n converges to 0, in probability.

Problem 7. Suppose that a sequence X_n of random variables satisfies

$$\lim_{n \rightarrow \infty} \mathbf{E}[|X_n - c|^\alpha] = 0,$$

where α is a positive number. Show that the sequence X_n converges to c in probability.

Solution. Using the Markov inequality, we have

$$\mathbf{P}(|X_n - c| \geq \epsilon) = \mathbf{P}(|X_n - c|^\alpha \geq \epsilon^\alpha) \leq \frac{\mathbf{E}[(X_n - c)^\alpha]}{\epsilon^\alpha}.$$

Taking the limit as $n \rightarrow \infty$, we obtain

$$\lim_{n \rightarrow \infty} \mathbf{P}(|X_n - c| \geq \epsilon) = 0,$$

which establishes convergence in probability.

Problem 8. Consider a Poisson process, with mean arrival rate $\lambda = 1$, and let X_n be the number of arrivals between time zero and n . Does X_n/n converge in probability?

Solution. Let A_i be the number of arrivals in the interval $[i, i + 1)$. The random variables A_i are independent, identically distributed, with finite variance. We have $X_n = A_1 + \cdots + A_n$, so that X_n/n is the sample mean of the A_i . By the weak law of large numbers, X_n/n converges to zero, in probability.

Problem 9. Let $X_1, X_2, \dots$ be independent, identically distributed random variables with (unknown but finite) mean μ and positive variance. For $i = 1, 2, \dots$, let

$$Y_i = \frac{1}{3}X_i + \frac{2}{3}X_{i+1}.$$

- (a) Are the random variables Y_i independent?
- (b) Are they identically distributed?
- (c) Let

$$M_n = \frac{1}{n} \sum_{i=1}^n Y_i.$$

Show that M_n converges to μ in probability.

Solution. (a) No. For example, suppose that the X_i have zero mean. Then,

$$\mathbf{E}[Y_1 Y_2] = \mathbf{E} \left[\left(\frac{1}{3}X_1 + \frac{2}{3}X_2 \right) \left(\frac{1}{3}X_2 + \frac{2}{3}X_3 \right) \right] = \frac{2}{9} \cdot \mathbf{E}[X_2^2] \neq 0 = \mathbf{E}[Y_1] \cdot \mathbf{E}[Y_2].$$

(b) Yes, because the joint distribution of X_i and X_{i+1} is the same for all i .

(c) Note that $\mathbf{E}[M_n] = \mu$. One approach to the problem is to calculate the variance of M_n and show that it converges to zero. Using the Chebyshev inequality, the convergence of M_n to μ follows.

A simpler approach is to note that

$$M_n = \frac{1}{3} \cdot \frac{1}{n} \sum_{i=1}^n X_i + \frac{2}{3} \cdot \frac{1}{n} \sum_{i=1}^n X_{i+1}.$$

By the weak law of large numbers the first sum converges to $\mu/3$ and the second to $2\mu/3$, in probability. This readily implies that M_n converges to μ , in probability. (Rigorously speaking, this makes use of the result of Problem 4 in Chapter 7 of the text: if two sequences of random variables converge in probability, then their sum also converges in probability to the sum of the two limits.)

Problem 10. Let $X_1, X_2, \dots$ be a sequence of independent random variables that are uniformly distributed between 0 and 1. For every n , we let Y_n be the median of the values of $X_1, X_2, \dots, X_{2n+1}$. [That is, we order $X_1, \dots, X_{2n+1}$ in increasing order and let Y_n be the $(n+1)$ st element in this ordered sequence.] Show that the sequence Y_n converges to $1/2$, in probability.

Solution. Let us fix some $\epsilon > 0$. We will show that $\mathbf{P}(Y_n \geq 0.5 + \epsilon)$ converges to 0. By symmetry, this will imply that $\mathbf{P}(Y_n \leq 0.5 - \epsilon)$ also converges to zero, and it will follow that Y_n converges to 0.5, in probability.

For the event $\{Y_n \geq 0.5 + \epsilon\}$ to occur, we must have at least $n+1$ of the random variables $X_1, X_2, \dots, X_{2n+1}$ to have a value of $0.5 + \epsilon$ or larger. Let Z_i be a Bernoulli random variable which is equal to 1 if and only if $X_i \geq 0.5 + \epsilon$. Then, the event $\{Y_n \geq 0.5 + \epsilon\}$ is the same as the event $(Z_1 + \dots + Z_{2n+1})/(2n+1) \geq 0.5$. Note that $\mathbf{P}(Z_i = 1) = 0.5 - \epsilon$. By the weak law of large numbers, the sequence $(Z_1 + \dots + Z_{2n+1})/(2n+1)$ converges to $0.5 - \epsilon$. Therefore,

$$\mathbf{P}\left(\frac{Z_1 + \dots + Z_{2n+1}}{2n+1} \geq 0.5\right)$$

converges to zero, which implies that $\mathbf{P}(Y_n \geq 0.5 + \epsilon)$ also converges to zero.

SECTION 7.4. The Central Limit Theorem

Problem 11. Uncle Henry has been having trouble keeping his weight constant. In fact, at the end of each week, he notices that his weight has changed by a random amount, uniformly distributed between -0.5 and 0.5 pounds. Assuming that the weight change during any given week is independent of the weight change of any other week, find the probability that Uncle Henry will gain or lose more than 3 pounds in the next 50 weeks.

Solution. We will use the Central Limit Theorem to compute the required probability. Let X_i be the amount Uncle Henry gains or loses in the i th week. We have

$$\begin{aligned}\mathbf{P}\left(\left|\sum_{i=1}^{50} X_i\right| \geq 3\right) &= 2\mathbf{P}\left(\sum_{i=1}^{50} X_i \geq 3\right) \\ &= 2 - 2\mathbf{P}\left(\sum_{i=1}^{50} X_i \leq 3\right) \\ &= 2 - 2\mathbf{P}\left(\frac{\sum_{i=1}^{50} X_i - 50 \cdot 0}{\sigma\sqrt{50}} \leq \frac{3 - 50 \cdot 0}{\sigma\sqrt{50}}\right) \\ &= 2 - 2\mathbf{P}(Z \leq 1.4697) \\ &\approx 0.1416,\end{aligned}$$

where Z is a standard normal random variable.

Problem 12. On any given flight, an airline's goal is to fill the plane as much as possible, without overbooking. If, on average, 10% of customers cancel their tickets, all independently of each other, what is the probability that a particular flight will be overbooked if the airline sells 320 tickets, for a plane that has maximum capacity 300 people? What is the probability that a plane with maximum capacity 150 people will be overbooked if the airline sells 160 tickets?

Solution. Let N be the number of people that show up. The flight will be overbooked if $N > 300$. We can use the normal approximation to the binomial to calculate this probability:

$$\mathbf{P}(301 \leq N \leq 320) = \Phi\left(\frac{320 - 288}{5.367}\right) - \Phi\left(\frac{301 - 288}{5.367}\right) = 1 - \Phi(2.4181) = 0.0078.$$

A similar computation for the smaller plane shows that the probability that the smaller plane will be overbooked is

$$\mathbf{P}(151 \leq N \leq 160) = 1 - \Phi\left(\frac{151 - 144}{3.794}\right) = 0.0329.$$

Problem 13. Alex puts some pennies into a piggy-bank each day. The number of pennies added on any given day is equally likely to be 1, 2, 3, 4, 5, or 6, and is independent from day to day. Find an approximation to the probability that it takes at least 80 days to collect 3 dollars.

Solution. Let X_n be the number of pennies added on the n th day. We are interested in the probability that $\sum_{i=1}^{80} X_i \leq 300$). Using the central limit theorem approximation, we have

$$\begin{aligned} \mathbf{P}\left(\sum_{i=1}^{80} X_i \leq 300\right) &= \mathbf{P}\left(\frac{\sum_{i=1}^{80} X_i - 80 \cdot 3.5}{\sqrt{(91/6)80}} \leq \frac{300 - 80 \cdot 3.5}{\sqrt{(91/6)80}}\right) \\ &\approx \mathbf{P}(Z \leq 0.5748) \\ &\approx 0.7157. \end{aligned}$$

Problem 14. Let $X_1, \dots, X_{10}$ be independent random variables, uniformly distributed over the unit interval $[0, 1]$.

- (a) Estimate $\mathbf{P}(X_1 + \dots + X_{10} \geq 7)$ using the Markov inequality.
- (b) Repeat part (a) using the Chebyshev inequality.
- (c) Repeat part (a) using the central limit theorem.

Solution. (a) To use the Markov inequality, let $X = \sum_{i=1}^{10} X_i$. Then,

$$\mathbf{E}[X] = 10\mathbf{E}[X_i] = 5,$$

and the Markov inequality yields

$$\mathbf{P}(X \geq 7) \leq \frac{5}{7} = 0.7142.$$

(b) Using the Chebychev inequality, we find that

$$\mathbf{P}(X \geq 7) \leq \frac{1/3}{1/3 + 4} = 0.076.$$

(c) Finally, using the Central Limit Theorem, we find that

$$\begin{aligned} \mathbf{P}\left(\sum_{i=1}^{10} X_i \geq 7\right) &= 1 - \mathbf{P}\left(\sum_{i=1}^{10} X_i \leq 7\right) \\ &= 1 - \mathbf{P}\left(\frac{\sum_{i=1}^{10} X_i - 5}{\sqrt{10/3}} \leq \frac{7 - 5}{\sqrt{10/3}}\right) \\ &= 0.1379. \end{aligned}$$

Problem 15. Let S_n be the number of successes in n independent Bernoulli trials, where the probability of success in each trial is $p = 1/2$. Provide a numerical value for the limit as n tends to infinity for each of the following three expressions.

(a) $\mathbf{P}\left(\frac{n}{2} - 10 \leq S_n \leq \frac{n}{2} + 10\right)$.

(b) $\mathbf{P}\left(\frac{n}{2} - \frac{n}{10} \leq S_n \leq \frac{n}{2} + \frac{n}{10}\right)$.

(c) $\mathbf{P}\left(\frac{n}{2} - \frac{\sqrt{n}}{2} \leq S_n \leq \frac{n}{2} + \frac{\sqrt{n}}{2}\right)$.

Solution. (a) The central limit theorem suggests that

$$\mathbf{P}\left(\frac{n}{2} - 10 \leq S_n \leq \frac{n}{2} + 10\right) \approx \Phi\left(\frac{20}{\sqrt{n}}\right) - \Phi\left(-\frac{20}{\sqrt{n}}\right),$$

which converges to zero as $n \rightarrow \infty$.

(b) The limit is one, by the weak law of large numbers.

(c) By the central limit theorem,

$$\mathbf{P}\left(\frac{n}{2} - \frac{\sqrt{n}}{2} \leq S_n \leq \frac{n}{2} + \frac{\sqrt{n}}{2}\right) \approx \Phi(1) - \Phi(-1) = 0.6826.$$

Problem 16. The adult population of Nowhereville consists of 300 males and 196 females. Each male (respectively, female) has a probability of 0.4 (respectively, 0.5) of casting a vote in the local elections, independently of everyone else. Find a good numerical approximation for the probability that more males than females cast a vote.

Solution. Let M and F be the number of males and females, respectively, that cast a vote. We need to find $\mathbf{P}(M > N)$, i.e., $\mathbf{P}(M - N > 0)$. The central limit theorem does not apply directly to the random variable $M - N$. However, the central limit theorem implies that M and N are well approximated by normal random variables. So, $M - N$ is the difference of two independent approximately normal random variables. Since the difference of two normal random variables is itself normal, it follows that $M - N$ is approximately normal.

The mean and variance of $M - N$ are found by

$$\mathbf{E}[M - N] = 300 \cdot 0.4 + 196 \cdot 0.5 = 120 - 98 = 22,$$

$$\text{var}(M - N) = \text{var}(M) + \text{var}(N) = 300 \cdot 0.4 \cdot 0.6 + 196 \cdot 0.5 \cdot 0.5 = 121.$$

Thus, the standard deviation of $M - N$ is 11. Let Z be a standard normal random variable. Using the central limit theorem approximation, we obtain

$$\begin{aligned} \mathbf{P}(M - N > 0) &= \mathbf{P}\left(\frac{M - N - 22}{11} > -\frac{22}{11}\right) \\ &\approx \mathbf{P}(Z \geq -2) \\ &= 0.9772. \end{aligned}$$

A slightly more refined estimate is obtained by expressing the event of interest as $\mathbf{P}(M - N \geq 1/2)$. We then have

$$\begin{aligned} \mathbf{P}(M - N \geq 1/2) &= \mathbf{P}\left(\frac{M - N - 22}{11} \geq -\frac{21.5}{11}\right) \\ &\approx \mathbf{P}(Z \geq -1.95) \\ &= 0.974. \end{aligned}$$

SECTION 7.5. The Strong Law of Large Numbers

Problem 17. Let $X_1, X_2, \dots$ be a sequence of independent identically distributed random variables with finite mean and variance. Show that the sequence $Y_n = X_n/n$ converges to zero, with probability 1.

Solution. Let $c = \mathbf{E}[X_i^2]$, which has been assumed to be finite. We observe that

$$\mathbf{E} \left[\sum_{n=1}^{\infty} Y_n^2 \right] = \sum_{n=1}^{\infty} \frac{c}{n^2} < \infty.$$

(See Problem 14 in Chapter 7 of the text for a discussion of why the expectation and the summation can be interchanged.) Therefore, $\sum_{n=1}^{\infty} Y_n^2 < \infty$ (with probability 1). It follows that Y_n^2 converges to zero, with probability 1, which implies that Y_n also converges to zero, with probability 1.

For a second approach, let $M_n = (X_1 + \dots + X_n)/n$. From the strong law of large numbers, we know that M_n converges to the mean μ of the X_i , with probability 1, and so does $M_{n-1} = (X_1 + \dots + X_{n-1})/(n-1)$. Note that

$$M_n \cdot \frac{n}{n-1} = \frac{X_1 + \dots + X_n}{n} \cdot \frac{n}{n-1}.$$

Since $n/(n-1)$ converges to 1, we obtain that $nM_n/(n-1)$ also converges to μ . We now observe that

$$M_{n-1} - M_n \cdot \frac{n}{n-1} = \frac{X_n}{n-1}.$$

Since the two sequences on the left-hand side converge to μ , with probability 1, the difference $X_n/(n-1)$ must also converge to zero with probability 1, from which it follows that X_n/n converges to zero with probability 1.

Problem 18. Let $X_1, X_2, \dots$ be independent identically distributed random variables with mean 5, variance 9, and such that $\mathbf{P}(X_n = 0) > 0$. For each of the following ways of defining Z_n , determine whether the sequence Z_n converges with probability 1, and if it does, identify the limit.

- (a) $Z_n = (X_1 + \dots + X_n)/n$.
- (b) $Z_n = (X_1 + \dots + X_n - 5n)/\sqrt{n}$.
- (c) $Z_n = (X_1^2 + \dots + X_n^2)/n$.
- (d) $Z_n = X_1 X_2 \dots X_n$.
- (e) $Z_n = (X_1 X_2 + X_2 X_3 + \dots + X_{n-1} X_n)/n$.

Solution. (a) Yes, by the strong law of large numbers.

(b) No. By the central limit theorem, the distribution of Z_n converges to a normal distribution.

(c) Yes, by the strong law of large numbers applied to the random variables X_i^2 .

(d) Yes. Since $\mathbf{P}(X_n = 0) > 0$, there will be some n for which $X_n = 0$, which implies that the sequence Z_n eventually settles to zero.

(e) Yes. Use the strong law of large numbers to argue that the two sequences $U_n = (X_1 X_2 + X_3 X_4 + \dots + X_{2n-1} X_{2n})/n$ and $V_n = (X_2 X_3 + X_4 X_5 + \dots + X_{2n} X_{2n+1})/n$ converge. The sequence Z_n is obtained by combining terms from the above two sequences, and must also converge. For example, we have $Z_{2n+1} = n(U_n + V_n)/(2n+1)$. Since $U_n + V_n$ converges, and since $n/(2n+1)$ converges, we obtain that Z_{2n+1} converges as n increases. A similar argument can be used to show that the even part, Z_{2n} , of the sequence also converges to the same limit.

Problem 19. The fortune X_n of a gambler evolves as $X_n = Z_n X_{n-1}$, where the Z_n are independent identically distributed random variables with PMF

$$p_Z(z) = \begin{cases} 1/3, & \text{for } z = 3, \\ 2/3, & \text{for } z = 1/3. \end{cases}$$

Assume that $X_0 = 1$.

- (a) Show that the expected fortune $\mathbf{E}[X_n]$ converges to infinity as n increases.
- (b) Show that the actual fortune X_n converges to 0 with probability 1.

Solution. (a) We have

$$\mathbf{E}[Z_i] = 3 \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = \frac{11}{9}.$$

Furthermore,

$$\mathbf{E}[X_n] = \mathbf{E}[Z_1 \cdots Z_n] = \mathbf{E}[Z_1] \cdots \mathbf{E}[Z_n] = \left(\frac{11}{9}\right)^n,$$

which converges to infinity.

(b) We have

$$\log_3 X_n = \log_3 + \sum_{i=1}^n \log_3 Z_i$$

and

$$\mathbf{E}[\log_3 Z_i] = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot (-1) = -\frac{1}{3}.$$

By the strong law of large numbers, applied to the sequence $\log_3 Z_n$, the sequence $(\log_3 X_n)/n$ converges to $-1/3$, with probability 1. This implies that $\log_3 X_n$ converges to $-\infty$, with probability 1, and, therefore, X_n converges to zero.

What is happening here is the following. The random variable X_n approaches 0 with probability 1. However, at each time n there is a small (and diminishing with n) probability that X_n takes a very large value, which makes the expectation $\mathbf{E}[X_n]$ grow to infinity.

Problem 20. Consider a Poisson process with arrival rate $\lambda = 1$. We divide the infinite time axis into disjoint intervals of length 1, $1/2$, $1/3$, $1/4$, etc., so that the k th interval has length $1/k$. These intervals are getting smaller, but their total length is infinite because $\sum_{k=1}^{\infty} 1/k = \infty$. Let Z_k be the number of arrivals during the k th interval. Does the sequence Z_k converge in probability (and if so, to what)? Does the sequence Z_k converge with probability 1 (and if so, to what)? Provide a brief justification.

Solution. We note that Z_k is a Poisson random variable with parameter $1/k$. Using the Markov inequality,

$$\mathbf{P}(|Z_k| \geq \epsilon) \leq \frac{\mathbf{E}[Z_k]}{\epsilon} = \frac{1}{k\epsilon}.$$

For every $\epsilon > 0$, this converges to 0, and therefore Z_k converges to zero in probability.

However, the sequence Z_k does not converge with probability 1. Indeed, since arrivals keep occurring as time goes to infinity, it follows that for every k_0 there is always a $k > k_0$ such that $Z_k > 0$.