INTRODUCTION TO PROBABILITY

by

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CHAPTER 1

ADDITIONAL PROBLEM SOLUTIONS

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Problem 1.

We have

$$P(B) = 1 - P(B^c) = 1 - 0.35 = 0.65.$$

Also, by rearranging the formula

$$P(A \cup B) = P(A) + P(B) - P(A \cap B),$$

we obtain

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.55 + 0.65 - 0.75 = 0.45.$$

Problem 2.

The set $A \cap (A \cup B)^c$ is always empty, because it can be written as $A \cap A^c \cap B^c$ or $\emptyset \cap B^c$, since $A \cap A^c = \emptyset$.

Problem 3.

(a) From de Morgan's laws, we have

$$A^c \cap B^c = (A \cup B)^c,$$

and by taking the complement of both sides the first equality follows. Also, from de Morgan's laws, we have

$$A^c \cup B^c = (A \cap B)^c,$$

and by taking the complement of both sides the second equality follows.

(b) We have

$$A = \{1, 3, 5\}, \qquad B = \{1, 2\}.$$

Thus,

$$A^c \cap B^c = \{2, 4, 6\} \cap \{3, 4, 5, 6\} = \{4, 6\},$$

$$(A^c \cap B^c)^c = \{1, 2, 3, 5\},$$

$$A \cup B = \{1, 2, 3, 5\},$$

so the first equality is verified. Similarly,

$$A^c \cup B^c = \{2, 4, 6\} \cup \{3, 4, 5, 6\} = \{2, 3, 4, 5, 6\},$$

$$(A^c \cup B^c)^c = \{1\},$$

$$A \cap B = \{1\},$$

so the second equality is verified.

Problem 4.

Let $N(S)$ denote the number of elements is a finite set. From the corresponding Venn Diagram [cf. Fig. 1.1(a)], we see that

$$N(A \cap B) = N(A) + N(B) - N(A \cup B).$$

Thus,

$$N(A \cap B) + N(A \cup B) = N(A) + N(B).$$

Problem 5.

We have $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B) = 1 - \mathbf{P}(A^c) + \mathbf{P}(B) - \mathbf{P}(A \cap B) = 1 - 0.6 + 0.3 - 0.2 = 0.5$.

Problem 6.

There are 12 possible outcomes,

$$(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3),$$

and each has probability 1/12. The required probabilities are obtained by counting the number of outcomes that satisfy the corresponding criterion and dividing by 12. The answers are: (a) 1/2. (b) 1/6. (c) 1/3.

Problem 7.

Let $\mathbf{P}(S = k) = ak$, where a is the constant of proportionality. Since the events $\{S = k\}$, for $k = 2, 3, \ldots, 8$, are disjoint and form a partition of the sample space, we have $\sum_{k=2}^{8} ak = 1$ or

$$a = \frac{1}{2 + 3 + \cdots + 8} = \frac{1}{35}.$$

The sums 2 and 8 can arise in only one way, (1,1) and (4,4), respectively, so

$$\mathbf{P}(1,1) = \mathbf{P}(S = 2) = \frac{2}{35}, \qquad \mathbf{P}(4,4) = \mathbf{P}(S = 8) = \frac{8}{35}.$$

The sums 3 and 7 can arise in two ways, so

$$\mathbf{P}(2,1) = \mathbf{P}(1,2) = \frac{\mathbf{P}(S = 3)}{2} = \frac{3}{2 \cdot 35}, \qquad \mathbf{P}(3,4) = \mathbf{P}(4,3) = \frac{\mathbf{P}(S = 7)}{2} = \frac{7}{2 \cdot 35}.$$

The sums 4 and 6 can arise in three ways, so

$$\mathbf{P}(1,3) = \mathbf{P}(2,2) = \mathbf{P}(3,1) = \frac{\mathbf{P}(S = 4)}{3} = \frac{4}{3 \cdot 35},$$

$$\mathbf{P}(2,4) = \mathbf{P}(3,3) = \mathbf{P}(4,2) = \frac{\mathbf{P}(S = 6)}{3} = \frac{6}{3 \cdot 35}.$$

The sum 5 can arise in four ways, so

$$\mathbf{P}(1,4) = \mathbf{P}(2,3) = \mathbf{P}(3,2) = \mathbf{P}(4,1) = \frac{\mathbf{P}(S = 5)}{4} = \frac{5}{4 \cdot 35}.$$

The probability of getting doubles is

$$\sum_{k=1}^{4} \mathbf{P}(k,k) = \frac{2}{35} + \frac{4}{3 \cdot 35} + \frac{6}{3 \cdot 35} + \frac{8}{35} = \frac{40}{105}.$$

Problem 8.

We claim that the optimal order is to play the weakest player second (the order in which the other two opponents are played makes no difference). To see this, let p_i be the probability of winning against the opponent played in the ith turn. Then you will win the tournament if you win against the 2nd player (prob. p_2) and also you win against at least one of the two other players [prob. $p_1 + (1 - p_1)p_3 = p_1 + p_3 - p_1p_3$]. Thus the probability of winning the tournament is

$$p_2(p_1 + p_3 - p_1p_3).$$

The order $(1, 2, 3)$ is optimal if and only if the above probability is no less than the probabilities corresponding to the two alternative orders:

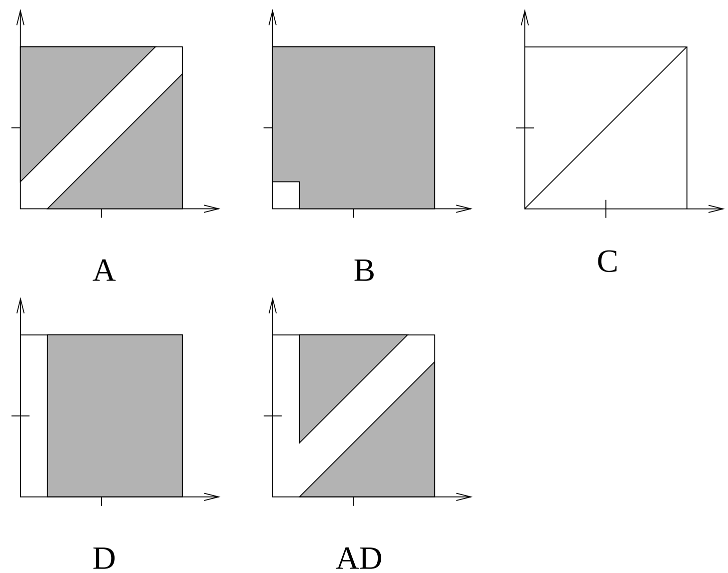
$$p_2(p_1 + p_3 - p_1p_3) \geq p_1(p_2 + p_3 - p_2p_3),$$

$$p_2(p_1 + p_3 - p_1p_3) \geq p_3(p_2 + p_1 - p_2p_1).$$

It can be seen that the first inequality above is equivalent to $p_2 \geq p_1$, while the second inequality above is equivalent to $p_2 \geq p_3$.

Problem 9.

Each of the events of interest can be described by a region in two-dimensional space, where the horizontal axis is Alice's number and the vertical axis is Bob's number; see the accompanying figure.



Note that the events A and $A \cap B$ are identical because A is a subset of B. We are given that the probability of any event is proportional to its area. Since the sample space has an area of 4, the probability of each event must be equal to its area divided by 4 to satisfy the normalization axiom. We then obtain

$$\mathbf{P}(A) = \frac{25}{36}, \quad \mathbf{P}(B) = \frac{35}{36}, \quad \mathbf{P}(A \cap B) = \frac{25}{36}, \quad \mathbf{P}(C) = 0, \quad \mathbf{P}(D) = \frac{5}{6}, \quad \mathbf{P}(A \cap D) = \frac{41}{72}.$$

Problem 10.

Since the events $A \cap B^c$ and $A^c \cap B$ are disjoint, we have using the additivity axiom repeatedly,

$$\mathbf{P}\big((A \cap B^c) \cup (A^c \cap B)\big) = \mathbf{P}(A \cap B^c) + \mathbf{P}(A^c \cap B) = \mathbf{P}(A) - \mathbf{P}(A \cap B) + \mathbf{P}(B) - \mathbf{P}(A \cap B).$$

Problem 11.

(a) We use the formulas $\mathbf{P}(A\cup B)=\mathbf{P}(A)+\mathbf{P}(A^c\cap B)$ and $(A\cup B\cup C)^c=A^c\cap B^c\cap C^c$. We have

$$\begin{aligned}
\mathbf{P}(A\cup B\cup C\cup D) &= \mathbf{P}(A\cup B\cup C)+\mathbf{P}((A\cup B\cup C)^c\cap D)\\
&= \mathbf{P}(A\cup B\cup C)+\mathbf{P}(A^c\cap B^c\cap C^c\cap D)\\
&= \mathbf{P}(A\cup B)+\mathbf{P}((A\cup B)^c\cap C)+\mathbf{P}(A^c\cap B^c\cap C^c\cap D)\\
&= \mathbf{P}(A\cup B)+\mathbf{P}(A^c\cap B^c\cap C)+\mathbf{P}(A^c\cap B^c\cap C^c\cap D)\\
&= \mathbf{P}(A)+\mathbf{P}(A^c\cap B)+\mathbf{P}(A^c\cap B^c\cap C)+\mathbf{P}(A^c\cap B^c\cap C^c\cap D).
\end{aligned}$$

(b) Use induction and verify the main induction step by emulating the derivation of part (a).

Problem 12.

Let A be the event that your friend searches disc 1 and finds nothing, and let B_i be the event that your thesis is on disc i. Now note that for $i \neq 1$, we have $\mathbf{P}(A \cap B_i) = \mathbf{P}(B_i)$, so that

$$\mathbf{P}(B_i \,|\, A) = \frac{\mathbf{P}(A \cap B_i)}{\mathbf{P}(A)} = \frac{\mathbf{P}(B_i)}{\mathbf{P}(A)} = \frac{0.25}{1 - 0.25 \cdot 0.4} = 0.2777.$$

For $i = 1$, we have

$$\mathbf{P}(B_i \,|\, A) = \frac{\mathbf{P}(A \cap B_i)}{\mathbf{P}(A)}.$$

Since $\mathbf{P}(A \,|\, B_i) = \mathbf{P}(A \cap B_i)/\mathbf{P}(B_i)$, we have

$$\mathbf{P}(A \cap B_i) = \mathbf{P}(B_i)\mathbf{P}(A \,|\, B_i) = 0.25(1 - 0.4).$$

Therefore,

$$\mathbf{P}(B_i \,|\, A) = \frac{0.25(1 - 0.4)}{1 - 0.25 \cdot 0.4} = 0.1666.$$

Problem 13.

The probability of dialing correctly with less or equal to k tries is 1 minus the probability of dialing incorrectly with k successive tries. The latter probability is

$$\frac{9}{10}\frac{8}{9}\cdots\frac{10-k}{11-k},$$

since $\frac{10-i}{11-i}$ is the conditional probability that the ith try is unsuccessful, given that the preceding $i-1$ tries are unsuccessful. Therefore, the desired number of tries is the smallest number $k \geq 1$ such that

$$\frac{9}{10}\frac{8}{9}\cdots\frac{10-k}{11-k} < 0.5.$$

Problem 14.

(a) There are 6 possible outcomes that are doubles, so the probability of doubles is $6/36 = 1/6$.

(b) The conditioning event (sum is 4 or less) consists of the 6 outcomes

$$\big\{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)\big\},$$

2 of which are doubles, so the conditional probability of doubles is $2/6 = 1/3$.

(c) There are 11 possible outcomes with at least one 6, namely, $(6,6)$, $(6,i)$, and $(i,6)$, for $i = 1,2,\ldots,5$. The probability that at least one die is a 6 is $11/36$.

(d) There are 30 possible outcomes where the dice land on different numbers. Out of these, there are 10 outcomes in which at least one of the rolls is a 6. Thus, the desired conditional probability is $10/30 = 1/3$.

Problem 15.

Let A be the event that the student is not overstressed, and let A^c be the event that the student is in fact overstressed. Now let B be the event that the test results indicate that the student is not overstressed. The desired probability, $\mathbf{P}(A \mid B)$, is found by Bayes' rule:

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A)\mathbf{P}(B \mid A)}{\mathbf{P}(A)\mathbf{P}(B \mid A) + \mathbf{P}(A^c)\mathbf{P}(B \mid A^c)} = \frac{0.005 \cdot 0.95}{0.005 \cdot 0.95 + 0.995 \cdot 0.15} \approx 0.03.$$

Problem 16.

Let A_i be the event corresponding to starting with trail i. We have

$$\mathbf{P}(A_i) = \frac{1}{n}, \qquad i = 1, 2, \ldots, n.$$

Let also B be the event of reaching the destination. We have

$$\mathbf{P}(B \mid A_i) = \frac{1}{1+i}, \qquad i = 1, 2, \ldots, n.$$

Thus, by the total probability theorem, the probability of reaching the destination is

$$\mathbf{P}(B) = \sum_{i=1}^{n} \mathbf{P}(A_i)\mathbf{P}(B \mid A_i) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1+i}.$$

Problem 17.

Let B be the event that Bob tossed more heads. Let X be the event that after each has tossed n of their coins, Bob has more heads than Alice, let Y be the event that under the same conditions, Alice has more heads than Bob, and let Z be the event that they have the same number of heads. Since the coins are fair, we have $\mathbf{P}(X) = \mathbf{P}(Y)$, and also $\mathbf{P}(Z) = 1 - \mathbf{P}(X) - \mathbf{P}(Y)$. Furthermore, we see that

$$\mathbf{P}(B \,|\, X) = 1, \qquad \mathbf{P}(B \,|\, Y) = 0, \qquad \mathbf{P}(B \,|\, Z) = \frac{1}{2}.$$

Now we have, using the theorem of total probability,

$$\begin{aligned}
\mathbf{P}(B) &= \mathbf{P}(X) \cdot \mathbf{P}(B \,|\, X) + \mathbf{P}(Y) \cdot \mathbf{P}(B \,|\, Y) + \mathbf{P}(Z) \cdot \mathbf{P}(B \,|\, Z) \\
&= \mathbf{P}(X) + \frac{1}{2} \cdot \mathbf{P}(Z) \\
&= \frac{1}{2} \cdot \big(\mathbf{P}(X) + \mathbf{P}(Y) + \mathbf{P}(Z)\big) \\
&= \frac{1}{2}.
\end{aligned}$$

as required. What is happening here is that Alice's probability of more heads than Bob is less than 1/2, so Bob has an advantage. However, the probability of equal number of heads is positive, and when added to Alice's probability of more heads, it gives 1/2.

Problem 18.

Let A_i be the event that bit i was on the tape and B_j be the event that bit j was detected. We want to find $\mathbf{P}(A_1 \mid B_1)$. Using Bayes' rule, we get

$$\begin{aligned}\mathbf{P}(A_1 \mid B_1) &= \frac{\mathbf{P}(A_1)\mathbf{P}(B_1 \mid A_1)}{\mathbf{P}(A_0)\mathbf{P}(B_1 \mid A_0) + \mathbf{P}(A_1)\mathbf{P}(B_1 \mid A_1)} \\ &= \frac{0.5 \cdot 0.85}{0.5 \cdot 0.1 + 0.5 \cdot 0.85} \\ &\approx 0.89.\end{aligned}$$

Problem 19.

The probability that the customer receives service 3 out of 4 times is given by the binomial formula:

$$\binom{4}{3}\cdot\left(\frac{3}{4}\right)^3\cdot\left(\frac{1}{4}\right)=\frac{27}{64}.$$

Problem 20.

Let p_i and q_i be respectively the probability and the area of face i. We have

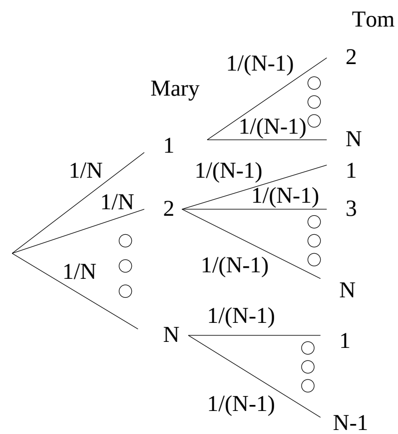
$$p_i = \frac{q_i}{\sum_{j=1}^{6} q_j}, \qquad i = 1, \ldots, 6.$$

Thus the probability that we get doubles is

$$\sum_{i=1}^{6} p_i^2 = \frac{\sum_{i=1}^{6} q_i^2}{\left(\sum_{j=1}^{6} q_j\right)^2} = \frac{2(1.5)^2 + 2(0.4)^2 + 2(0.4 \cdot 1.5)^2}{\left(2(1.5) + 2(0.4) + 2(0.4 \cdot 1.5)\right)^2} = 0.2216.$$

Problem 21.

For convenience, we will number each of the parking spaces. We will draw a sequential probability tree to illustrate the sample space:



Mary can choose any of the n parking spaces. She has a probability of $1/n$ of selecting any particular space. Tom can choose any of the remaining $n-1$ spaces and has a probability of $1/(n-1)$ of choosing any particular space (other than the one Mary chose).

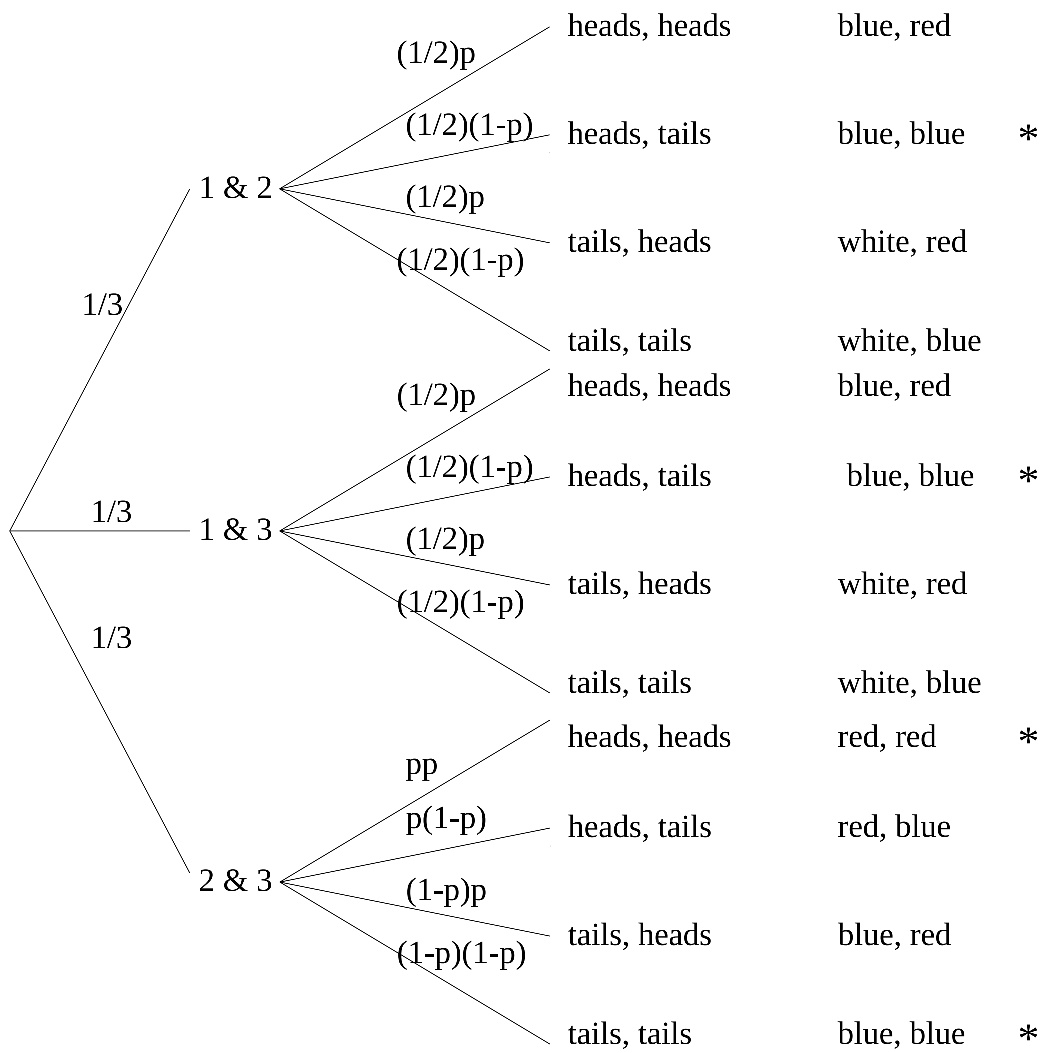
There are $n(n-1)$ leaves on the tree, and each leaf is equally likely to occur. When we look at the leaves on the branches where Mary does not choose spaces $1, 2, n-1$, or n, we see that 4 leaves on each of these branches is in our event (two spaces on each side of Mary's car). When Mary chooses spaces 2 or $n-1$, there are three such leaves (one space on one side and two on the other side). When Mary chooses spaces 1 or n, there are only two such leaves (she is at one end or the other of the parking lot). Therefore, the probability that they are parked within two spaces of each other is:

$$\mathbf{P}(A) = \frac{(4)(n-4) + (3)(2) + (2)(2)}{n(n-1)} = \frac{4n-6}{n(n-1)}$$

Note that for $n = 2$ and $n = 3$, $\mathbf{P}(A) = 1$, as expected.

Problem 22.

There are two stages to the experiment: the selection of the two coins and the tossing of the coins. There are three different ways that two coins can be selected: the 1st and 2nd, the 1st and 3rd, and the 2nd and 3rd. Each of these pairs are equally likely to be selected. For each pair, the tosses have four possible outcomes: (heads, heads), (heads, tails), (tails, heads), (tails, tails). The sample space can be described as in the figure:



The probability that the sides that land face up are the same is then:

$$\begin{aligned}
\mathbf{P}(\text{same}) &= \mathbf{P}\big((\text{blue,blue})\big) + \mathbf{P}\big((\text{red,red})\big) \\
&= \frac{1}{3}\left\{\frac{1}{2}(1-p) + \frac{1}{2}(1-p) + p^2 + (1-p)^2\right\} \\
&= \frac{1}{3}(2p^2 - 3p + 2).
\end{aligned}$$

Setting this equal to $\frac{29}{96}$ and solving the quadratic equation for p yields $p = 5/8$ or $p = 7/8$.

Problem 23.

Let Q be the event that someone is qualified, Q^c the event that someone is unqualified, and A the event that the 20 questions correctly determine whether the candidate is qualified or not. Using the total probability theorem, we have

$$\begin{aligned}\mathbf{P}(A) &= \mathbf{P}(Q)\mathbf{P}(A \,|\, Q) + \mathbf{P}(Q^c)\mathbf{P}(A \,|\, Q^c) \\ &= q\sum_{i=15}^{20}\binom{20}{i}p^i(1-p)^{20-i} + (1-q)\sum_{i=6}^{20}\binom{20}{i}p^i(1-p)^{20-i}.\end{aligned}$$

Problem 24.

The number of jurors that make the "wrong" decision can be modeled as a binomial random variable with parameters $n = 7$ and $p = .2$. The jury as a whole will make the wrong decision if 4, 5, 6, or 7 jurors make the wrong decision. Denote these events by A, B, C, D, respectively. Since these events are mutually exclusive, the probability of their union is the sum of their probabilities, so

$$\begin{aligned}
\mathbf{P}(\text{Jury Error}) &= \mathbf{P}(A) + \mathbf{P}(B) + \mathbf{P}(C) + \mathbf{P}(D) \\
&= \binom{7}{4}(0.2)^4 \cdot (0.8)^3 + \binom{7}{5}(0.2)^5 \cdot (0.8)^2 \\
&\quad + \binom{7}{6}(0.2)^6 \cdot (0.8)^1 + \binom{7}{7}(0.2)^7 \cdot (0.8)^0 \\
&= 0.033.
\end{aligned}$$

Problem 25.

The probability that persons 1 and 2 both roll a particular face is $1/n^2$. Therefore,

$$\mathbf{P}(A_{12}) = \mathbf{P}(A_{13}) = \mathbf{P}(A_{23}) = \frac{n}{n^2} = \frac{1}{n}.$$

Similarly, we also have

$$\mathbf{P}(A_{12} \cap A_{13}) = \mathbf{P}(\text{all players roll the same face}) = \frac{n}{n^3} = \frac{1}{n^2},$$

so $\mathbf{P}(A_{12} \cap A_{13}) = \mathbf{P}(A_{12}) \cdot \mathbf{P}(A_{13})$. Hence A_{12} and A_{13} are independent, and the same is true of any other pair from the events A_{12}, A_{13}, and A_{23}. However, A_{12}, A_{13}, and A_{23} are not independent. In particular, if A_{12} and A_{13} occur, then A_{13} also occurs.

Problem 26.

By definition, $\vee(A \,|\, B)$ is equal to

$$\frac{\mathbf{P}(A \,|\, B)}{\mathbf{P}(A^c \,|\, B)} = \frac{\mathbf{P}(A \cap B)/\mathbf{P}(B)}{\mathbf{P}(A^c \cap B)/\mathbf{P}(B)} = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A^c \cap B)} = \frac{\mathbf{P}(B \,|\, A)\mathbf{P}(A)}{\mathbf{P}(B \,|\, A^c)\mathbf{P}(A^c)} = \mathbf{L}(B \,|\, A) \vee (A).$$

Problem 27.

Let A be the event that May is lucky on the current day, and let $B_{m,n}$ be the event that m wins and n losses have occurred so far. We assume independence of the results of different spins/plays. Then we have, using the odds formula of the preceding problem and the binomial formula,

$$\frac{\vee(A \mid B_{m,n})}{\vee(A)} = L(B_{m,n} \mid A) = \frac{\mathbf{P}(B_{m,n} \mid A)}{\mathbf{P}(B_{m,n} \mid A^c)} = \frac{\binom{m+n}{m} p_L^m (1-p_L)^n}{\binom{m+n}{m} p_U^m (1-p_U)^n} = \left(\frac{p_L}{p_U}\right)^m \left(\frac{1-p_L}{1-p_U}\right)^n$$

From this formula, a convenient recursive algorithm is obtained. After $m+n$ plays, in which m wins and n losses occurred, we have

$$\vee(A \mid B_{m+1,n}) = \vee(A \mid B_{m,n}) \frac{p_L}{p_U}, \qquad \text{if she wins in the next play,}$$

$$\vee(A \mid B_{m,n+1}) = \vee(A \mid B_{m,n}) \frac{1-p_L}{1-p_U}, \qquad \text{if she loses in the next play.}$$

The initial condition is $\vee(A \mid B_{0,0})$ is equal to the initial (unconditional) odds $\vee(A)$ (which May knows by assumption).

Problem 28.

The events A and B are independent if and only if $\mathbf{P}(A)\mathbf{P}(B) = \mathbf{P}(A \cap B) = \mathbf{P}(A)$, where the last equality follows from the fact that $A \subset B$. This can be the case if and only if $\mathbf{P}(A) = 0$ or $\mathbf{P}(B) = 1$.

Problem 29.

The answer is no. Consider two tosses of a fair coin. The events $A = \{HH, TT\}$, $B = \{HH, HT\}$, $C = \{HH, TH\}$ satisfy the independence assumptions in the problem statement. On the other hand,

$$\mathbf{P}(A \mid B \cup C) = \frac{1}{3} \neq \frac{1}{2} = \mathbf{P}(A),$$

and A is not independent of $B \cup C$.

Problem 30.

To prove independence, we must show that

$$\mathbf{P}\big(A \cap (B \cup C)\big) = \mathbf{P}(A) \cdot \mathbf{P}(B \cup C).$$

Using the identity

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$

we obtain

$$\begin{aligned}
\mathbf{P}\big(A \cap (B \cup C)\big) &= \mathbf{P}\big((A \cap B) \cup (A \cap C)\big) \\
&= \mathbf{P}(A \cap B) + \mathbf{P}(A \cap C) - \mathbf{P}\big((A \cap B) \cap (A \cap C)\big) \\
&= \mathbf{P}(A \cap B) + \mathbf{P}(A \cap C) - \mathbf{P}(A \cap B \cap C).
\end{aligned}$$

The independence of A, B, and C implies that

$$\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B),$$

$$\mathbf{P}(A \cap C) = \mathbf{P}(A)\mathbf{P}(C),$$

$$\mathbf{P}(A \cap B \cap C) = \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C).$$

It follows that

$$\begin{aligned}
\mathbf{P}\big(A \cap (B \cup C)\big) &= \mathbf{P}(A)\mathbf{P}(B) + \mathbf{P}(A)\mathbf{P}(C) - \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C) \\
&= \mathbf{P}(A)\big(\mathbf{P}(B) + \mathbf{P}(C) - \mathbf{P}(B)\mathbf{P}(C)\big) \\
&= \mathbf{P}(A)\mathbf{P}(B \cup C).
\end{aligned}$$

Problem 31.

(a) Let A be the event that all 20 cars tested are good. We are asked to find $\mathbf{P}(K = 0 \,|\, A)$. Using Bayes' rule, we have

$$\mathbf{P}(K = 0 \,|\, A) = \frac{\mathbf{P}(K = 0)\mathbf{P}(A \,|\, K = 0)}{\sum_{i=0}^{9} \mathbf{P}(K = i)\mathbf{P}(A \,|\, K = i)}.$$

It is given that $\mathbf{P}(K = i) = 1/10$ for all i. To compute $\mathbf{P}(A \,|\, K = i)$, we condition on the event of exactly i lemons, and reason as follows. The first selected car has probability $(100-i)/100$ of being good. Having succeeded in the first selection, we are left with 99 cars out of which i are lemons; thus, the second selected car has probability $(99-i)/99$ of being good. Continuing similarly, and using the multiplication rule, we obtain

$$\mathbf{P}(A \,|\, K = i) = \frac{(100-i)(99-i)\cdots(81-i)}{100 \cdot 99 \cdot 81},$$

from which we can then obtain $\mathbf{P}(K = 0 \,|\, A)$.

(b) We use the exact same argument as in part (a), except that we need to recalculate $\mathbf{P}(A \,|\, K = i)$. Since the cars are chosen with replacement, we are dealing with 20 independent Bernoulli trials. The probability of finding a good car in any one trial is $(100-i)/100$. The probability of finding good cars in all 20 trials is

$$\mathbf{P}(A \,|\, k = i) = \left(\frac{100-i}{100}\right)^{20},$$

from which we can then obtain $\mathbf{P}(K = 0 \,|\, A)$.

Problem 32.

Think of lining up the jelly beans, by first placing the red ones, then the orange ones, etc. We also place 5 dividers to indicate where one color ends and another starts. (Note that two dividers can be adjacent if there are no jelly beans of some color.) By considering both jelly beans and dividers, we see that there is a total of 105 positions. Choosing the number of jelly beans of each color is the same as choosing the positions of the dividers. Thus, there are $\binom{105}{5}$ possibilities, and this is the number of possible jars.

Problem 33.

(a) We have

$$\mathbf{P}(A_n) = \binom{6n}{n} \left(\frac{1}{6}\right)^n \left(\frac{5}{6}\right)^{6n-n},$$

and

$$\mathbf{P}(B_n) = \sum_{k=n}^{6n} \binom{6n}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{6n-k}.$$

Both $\mathbf{P}(A_n)$ and $\mathbf{P}(B_n)$ change with n.

(b) $\mathbf{P}(A_n)$ converges to 0 as $n \to \infty$, while $\mathbf{P}(B_n)$ converges to a steady-state value as $n \to \infty$.

Problem 34.

(a) The number of outcomes that leads to a sum of 11 is the number of outcomes of the first two rolls that lead to a sum greater than or equal to $11 - 6 = 5$ and less than or equal to $11 - 1 = 10$. The number of outcomes of the first two rolls that leads to sum less than 5 or greater than 10 is 9. So the desired probability is

$$\frac{6^2 - 9}{6^3}.$$

(b) The number of outcomes that leads to a sum of 12 is the number of outcomes of the first two rolls that lead to a sum greater or equal to $12 - 6 = 6$ and less than or equal to $12 - 1 = 11$. The number of outcomes of the first two rolls that leads to sum less than 6 or greater than 11 is 11. So the desired probability is

$$\frac{6^2 - 11}{6^3}.$$

(c) Each of the sums 9 and 10 can be obtained in six distinct ways:

$$9 = 1 + 2 + 6 = 1 + 3 + 5 = 1 + 4 + 4 = 2 + 2 + 5 = 2 + 3 + 4 = 3 + 3 + 3,$$

$$10 = 1 + 3 + 6 = 1 + 4 + 5 = 2 + 2 + 6 = 2 + 3 + 5 = 2 + 4 + 4 = 3 + 3 + 4.$$

However, the number of outcomes that sum to 9 is 25, while the number of outcomes that sum to 10 is 27. Thus, a sum of 10 has probability $27/6^3$ and is more frequent than a sum of 9, which has probability $25/6^3$.

Problem 35.

(a) 2^{25}.

(b) First note that under this rule, each match will be stopped after a number of games ranging from 13 to 25. If a match will be stopped at the k'th game with player 1 having 13 points, then the last game was a win and $k-13$ of the previous games was a loss. So, there are $\binom{k-1}{k-13}$ matches that ends at the k'th game with player 1 having a score of 13. Taking player 2 into consideration and summing over k, we obtain

$$2\sum_{k=13}^{25}\binom{k-1}{k-13}$$

possible distinct score sequences.

Problem 36.

(a) The word "children" consists of 8 distinct letters, so the number of arrangements is the same as the number of possible permutations, namely 8!.
(b) In the word "bookkeeper", the letters "b", "o", "k", "e", "p", and "r" appear 1, 2, 2, 3, 1, and 1 times respectively. Arguing exactly as in the text the number of distinguishable rearrangements is

$$\frac{10!}{3!\,2!\,2!}.$$