A NEW LOOK AT CONVEX ANALYSIS AND OPTIMIZATION

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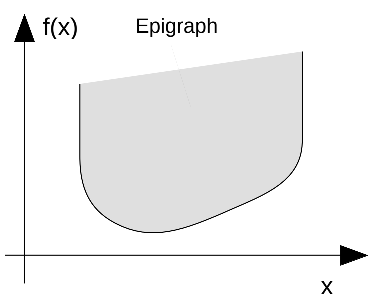
OUTLINE

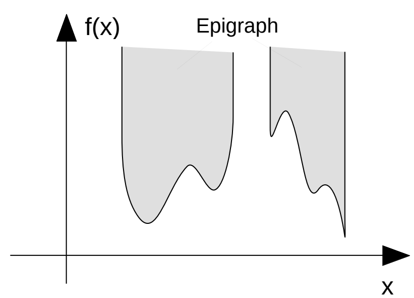
- **Convexity issues in optimization**
- **Historical remarks**
- **Our treatment of the subject**
- **Three unifying lines of analysis**
  - **Common geometrical framework for duality and minimax**
  - **Unifying framework for existence of solutions and duality gap analysis**
  - **Unification of Lagrange multiplier theory using an enhanced Fritz John theory and the notion of pseudonormality**

WHY IS CONVEXITY IMPORTANT IN OPTIMIZATION I

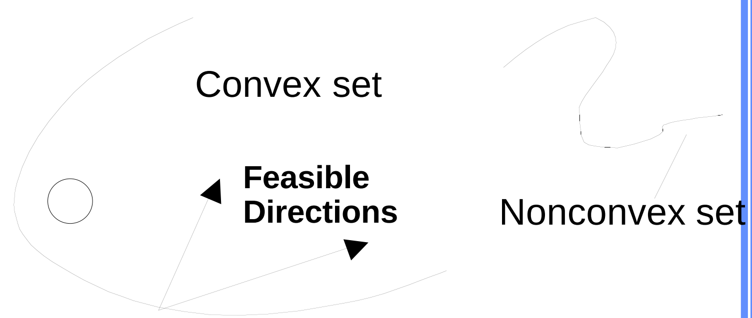
- **A convex function has no local minima that are not global**
- **A nonconvex function can be "convexified" while maintaining the optimality of its minima**
- **A convex set has nonempty relative interior**
- **A convex set has feasible directions at any point**



Convex function

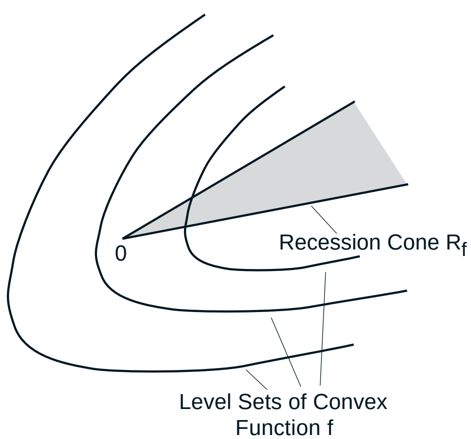


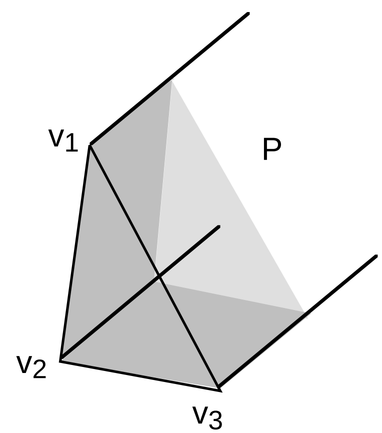
Nonconvex function



WHY IS CONVEXITY IMPORTANT IN OPTIMIZATION II

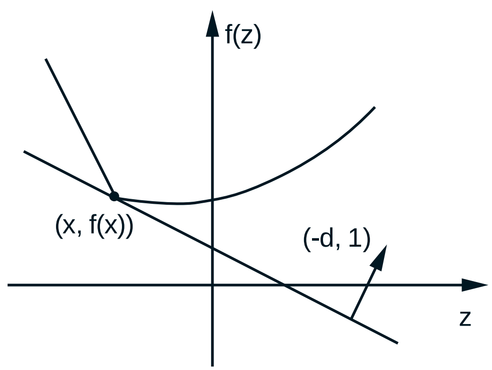
- The existence of minima of convex functions is conveniently characterized using directions of recession
- A polyhedral convex set is characterized by its extreme points and extreme directions

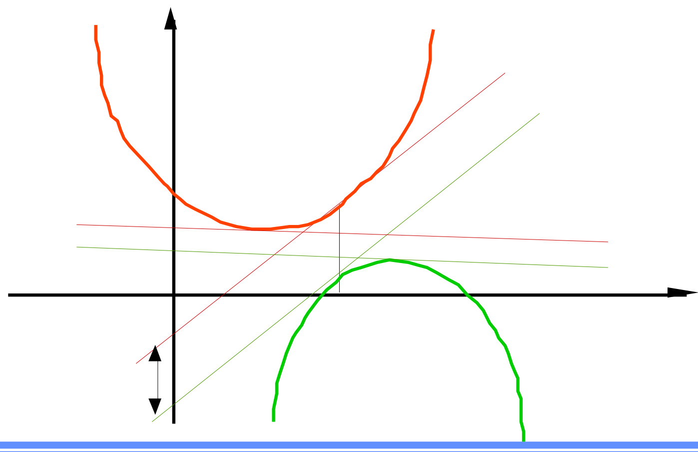




WHY IS CONVEXITY IMPORTANT IN OPTIMIZATION III

- **A convex function is continuous and has nice differentiability properties**
- **Convex functions arise prominently in duality**
- **Convex, lower semicontinuous functions are self-dual with respect to conjugacy**





SOME HISTORY

- **Late 19th-Early 20th Century:**
  - **Caratheodory, Minkowski, Steinitz, Farkas**
- **40s-50s: The big turning point**
  - **Game Theory: von Neumann**
  - **Duality: Fenchel, Gale, Kuhn, Tucker**
  - **Optimization-related convexity: Fenchel**
- **60s-70s: Consolidation**
  - **Rockafellar**
- **80s-90s: Extensions to nonconvex optimization and nonsmooth analysis**
  - **Clarke, Mordukovich, Rockafellar-Wets**

ABOUT THE BOOK

- **Convex Analysis and Optimization, by D. P. Bertsekas, with A. Nedic and A. Ozdaglar (March 2003)**
- **Aims to make the subject accessible through unification and geometric visualization**
- **Unification is achieved through several new lines of analysis**

NEW LINES OF ANALYSIS

I A unified geometrical approach to convex programming duality and minimax theory

- Basis: Duality between two elementary geometrical problems

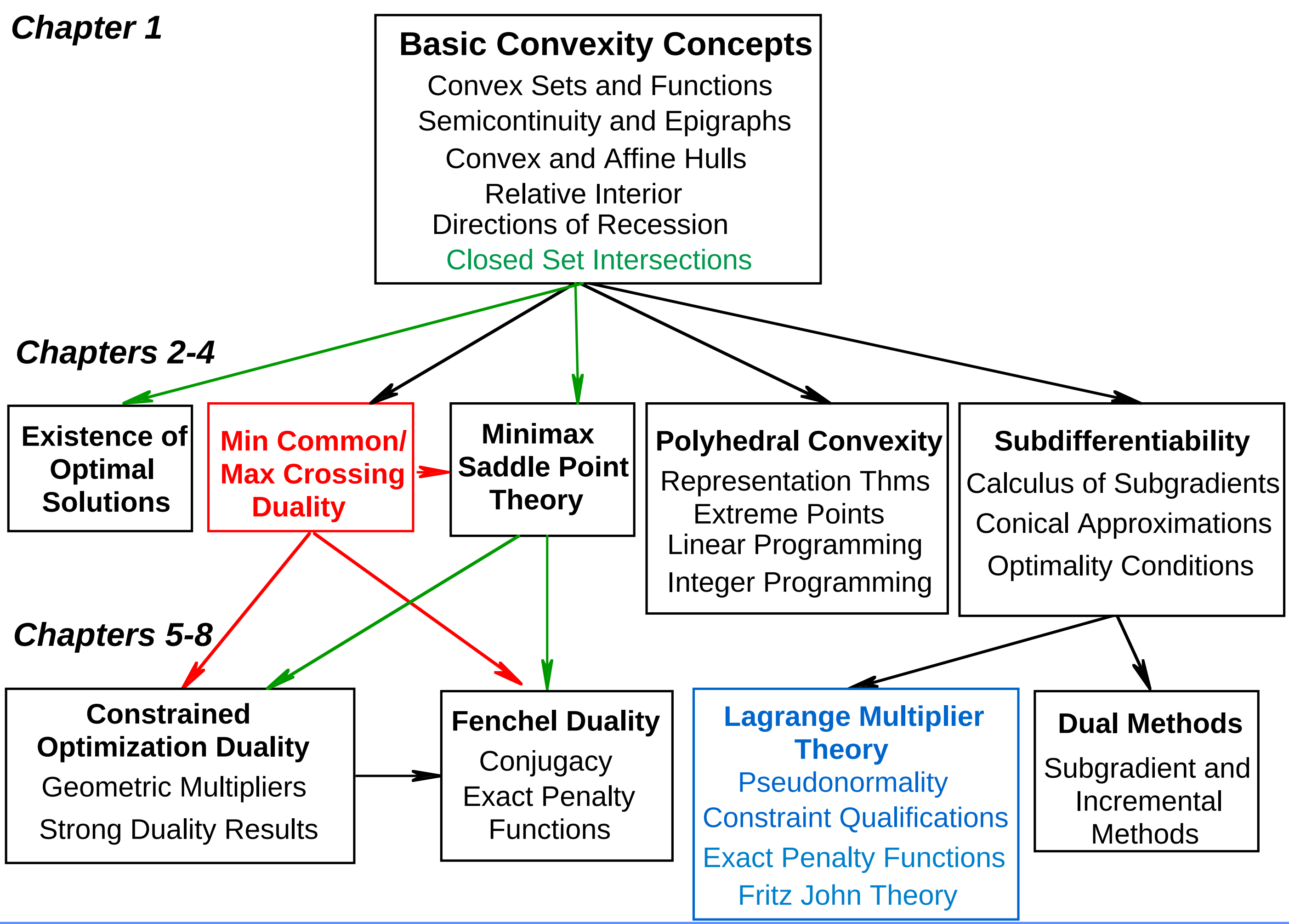
II A unified view of theory of existence of solutions and absence of duality gap

- Basis: Reduction to basic questions on intersections of closed sets

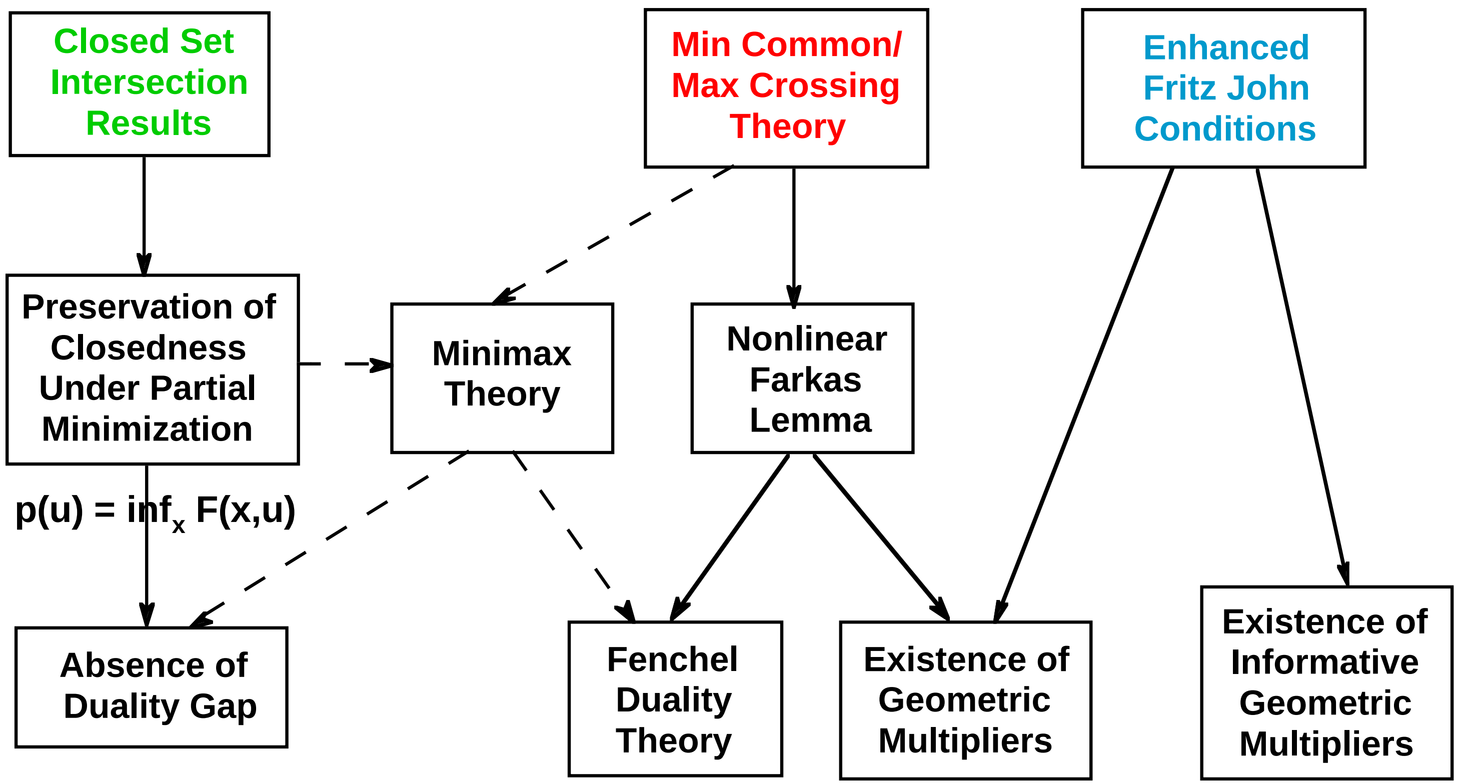
III A unified view of theory of existence of Lagrange multipliers/constraint qualifications

- Basis: The notion of constraint pseudonormality, motivated by a new set of enhanced Fritz John conditions

OUTLINE OF THE BOOK

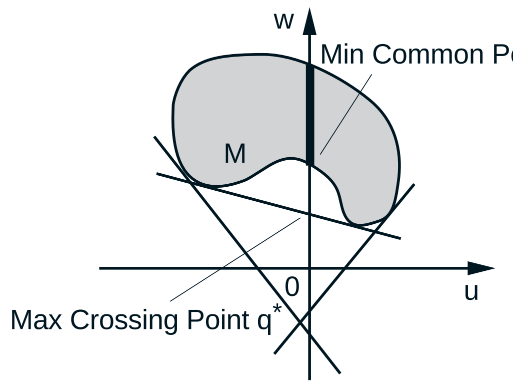


OUTLINE OF DUALITY ANALYSIS



I
MIN COMMON/MAX CROSSING DUALITY

GEOMETRICAL VIEW OF DUALITY



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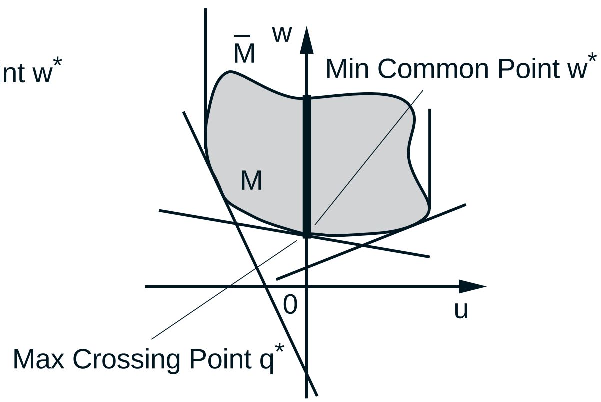

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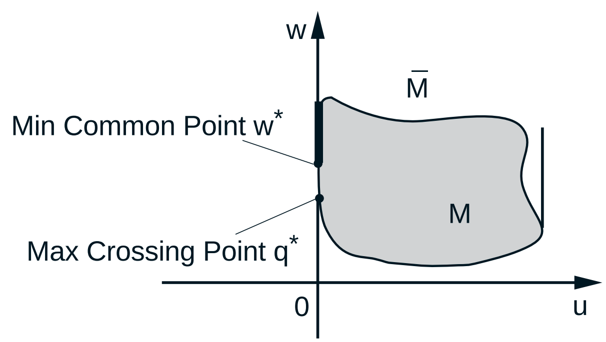

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APPROACH

- **Prove theorems about the geometry of M**
  - Conditions on M guarantee that $w^* = q^*$
  - Conditions on M that guarantee existence of a max crossing hyperplane
- **Choose M to reduce the constrained optimization problem, the minimax problem, and others, to special cases of the min common/max crossing framework**
- **Specialize the min common/max crossing theorems to duality and minimax theorems**

CONVEX PROGRAMMING DUALITY

- **Primal problem:**

  min $f(x)$ subject to $x \in X$ and $g_j(x) \leq 0$, $j=1,\ldots,r$

- **Dual problem:**

  max $q(\mu)$ subject to $\mu \geq 0$

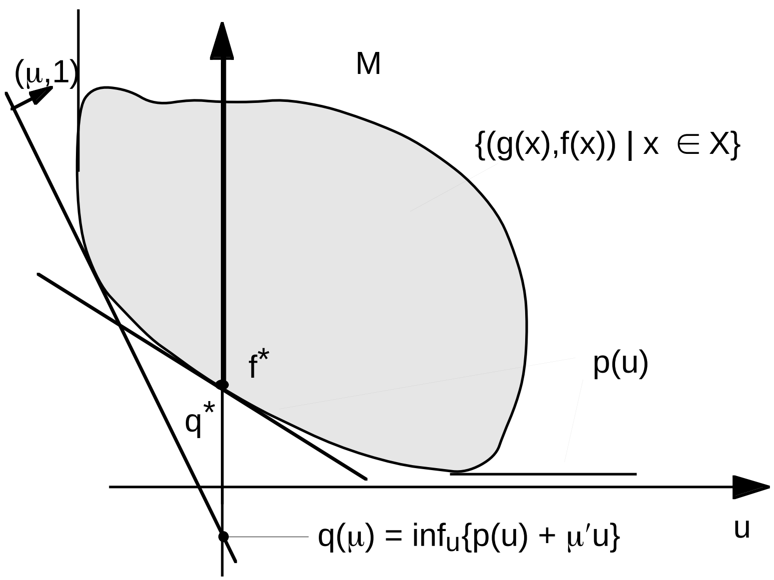
  where the dual function is

$$q(\mu) = \inf_x L(x,\mu) = \inf_x \{f(x) + \mu' g(x)\}$$

- **Optimal primal value = $\inf_x \sup_{\mu \geq 0} L(x,\mu)$**
- **Optimal dual value = $\sup_{\mu \geq 0} \inf_x L(x,\mu)$**
- **Min common/max crossing framework:**

$$M = \mathrm{epi}(p), \qquad p(u) = \inf_{x \in X,\, g_j(x) \leq u_j} f(x)$$

VISUALIZATION



MINIMAX / ZERO SUM GAME THEORY ISSUES

- Given a function $\Phi(x,z)$, where $x \in X$ and $z \in Z$, under what conditions do we have

$$\inf_x \sup_z \Phi(x,z) = \sup_z \inf_x \Phi(x,z)$$

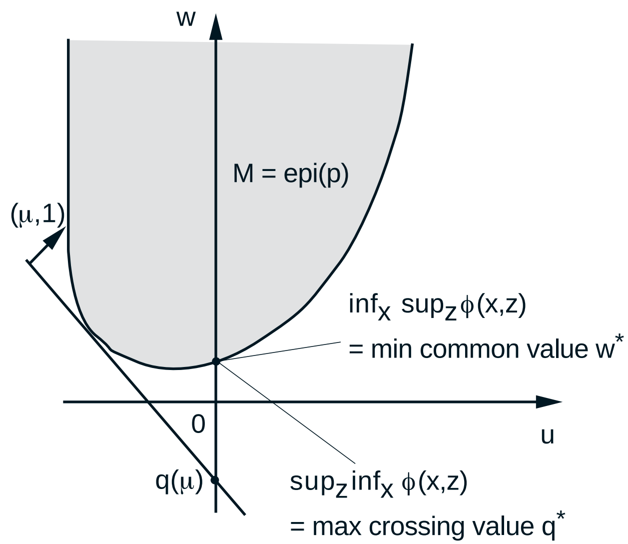
- Assume convexity/concavity, semicontinuity of Φ
- Min common/max crossing framework:

  M = epigraph of the function

$$p(u) = \inf_x \sup_z \{\Phi(x,z) - u'z\}$$

- $\inf_x \sup_z \Phi$ = Min common value
- $\sup_z \inf_x \Phi$ = Max crossing value

VISUALIZATION



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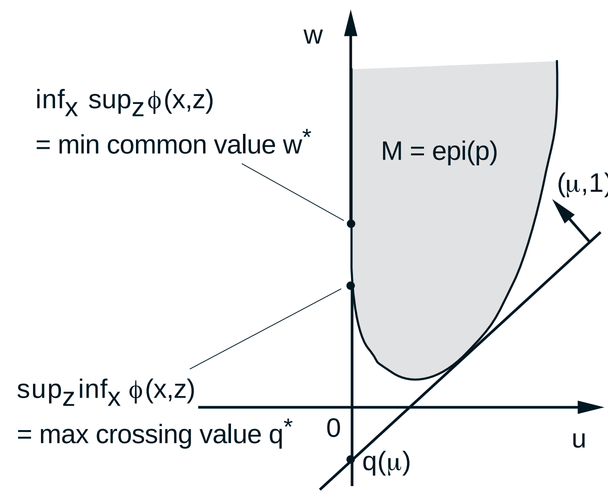

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TWO ISSUES IN CONVEX PROGRAMMING AND MINIMAX

- **When is there no duality gap (in convex programming), or inf sup = sup inf (in minimax)?**
- **When does an optimal dual solution exist (in convex programming), or the sup is attained (in minimax)?**
- **Min common/max crossing framework shows that**
  - **1st question is a lower semicontinuity issue**
  - **2nd question is an issue of existence of a nonvertical support hyperplane (or subgradient) at the origin**
- **Further analysis is needed for more specific answers**

II
UNIFICATION OF EXISTENCE AND NO DUALITY GAP ISSUES

INTERSECTIONS OF NESTED FAMILIES OF CLOSED SETS

- **Two basic problems in convex optimization**
  - **Attainment of a minimum of a function f over a set X**
  - **Existence of a duality gap**
- **The 1st question is a set intersection issue:**

  The set of minima is the intersection of the nonempty level sets $\{x \in X \mid f(x) \leq \gamma\}$
- **The 2nd question is a lower semicontinuity issue:**

  When is the function

  $$p(u) = \inf_x F(x,u)$$

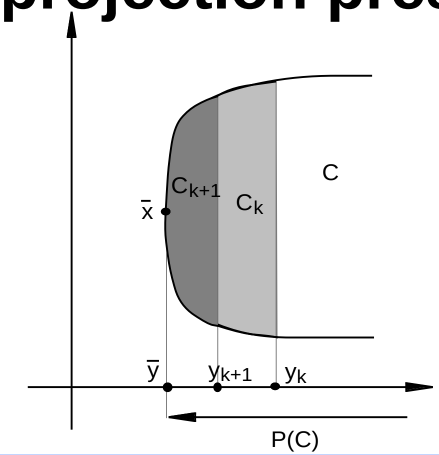
  lower semicontinuous, assuming F(x,u) is convex and lower semicontinuous?

PRESERVATION OF SEMICONTINUITY UNDER PARTIAL MINIMIZATION

- **2nd question also involves set intersection**
- **Key observation: For $p(u) = \inf_x F(x,u)$, we have**

$$\text{Closure}\big(P(\text{epi}(F))\big) \supset \text{epi}(p) \supset P\big(\text{epi}(F)\big)$$

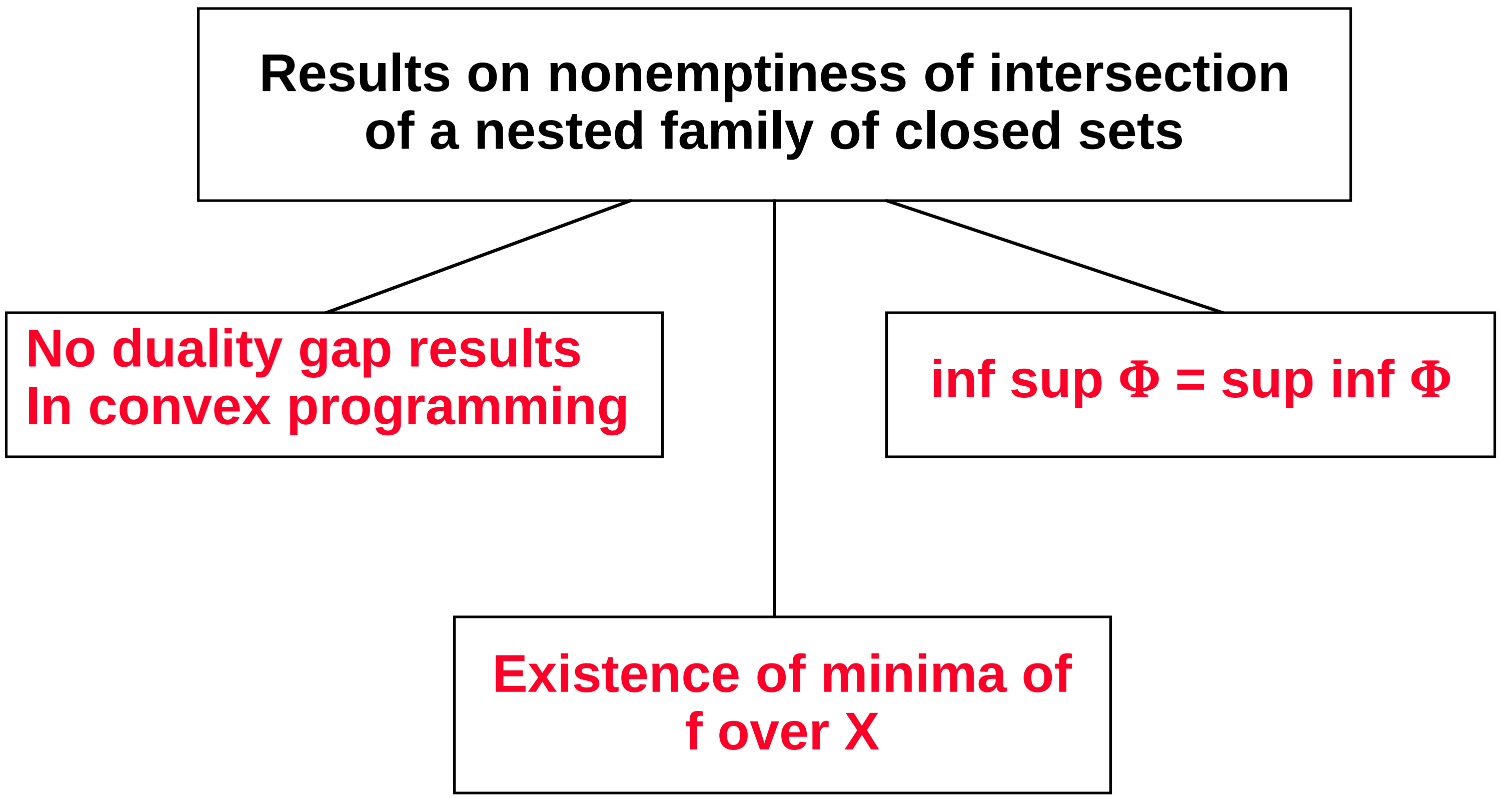
where $P(\cdot)$ is projection on the space of u. So if projection preserves closedness, F is l.s.c.



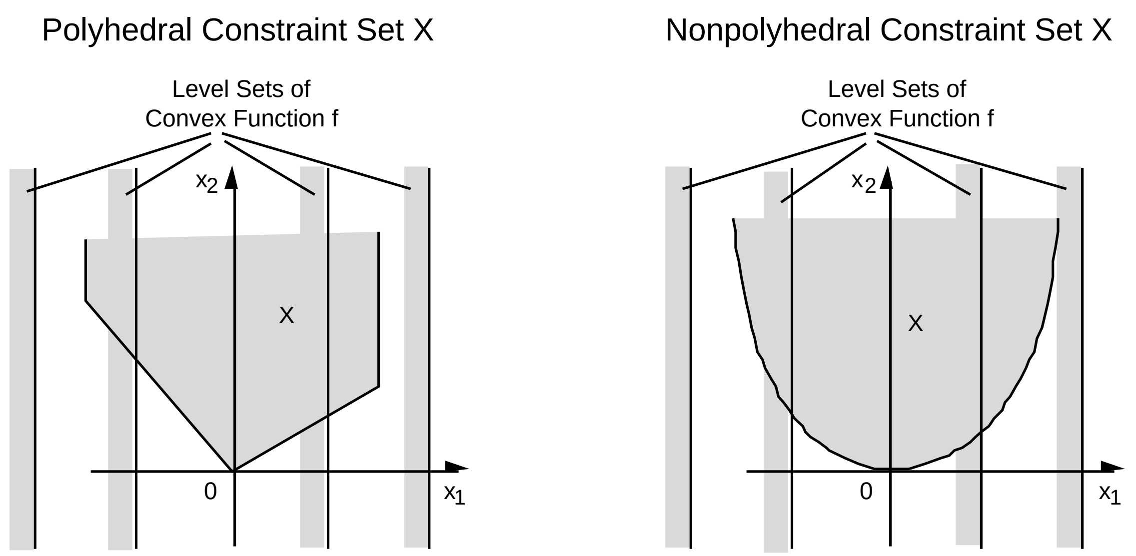
Given C, when is P(C) closed?

If y_k is a sequence in P(C) that converges to $\bar{y}$, we must show that the intersection of the C_k is nonempty

UNIFIED TREATMENT OF EXISTENCE OF SOLUTIONS AND DUALITY GAP ISSUES



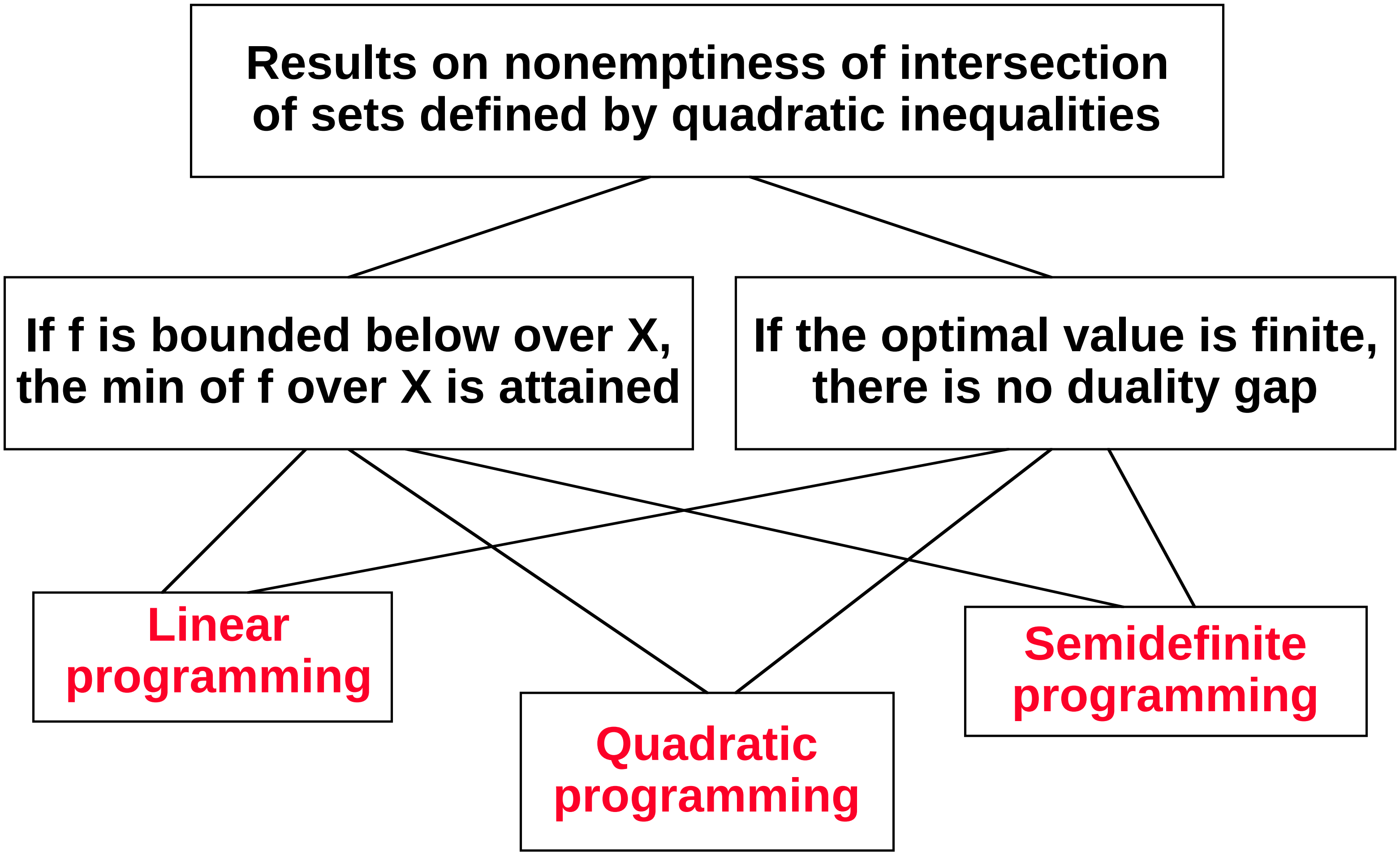
THE ROLE OF POLYHEDRAL ASSUMPTIONS: AN EXAMPLE



Minimum Attained $f(x) = e^{x_1}$ **Minimum not Attained**

The min is attained if X is polyhedral, and f is constant along common directions of recession of f and X

THE ROLE OF QUADRATIC FUNCTIONS



III
LAGRANGE MULTIPLIER THEORY / PSEUDONORMALITY

LAGRANGE MULTIPLIERS

- **Problem (smooth, nonconvex):**

  Min $f(x)$

  subject to $x \in X$, $h_i(x)=0$, $i = 1,\ldots,m$

- **Necessary condition for optimality of x^* (case $X = R^n$): Under some "constraint qualification", we have**

  $$\nabla f(x^*) + \Sigma_I \lambda_i \nabla h_i(x^*) = 0$$

  for some Lagrange multipliers λ_i

- **Basic analytical issue: What is the fundamental structure of the constraint set that guarantees the existence of a Lagrange multiplier?**

- **Standard constraint qualifications (case $X = R^n$):**
  - **The gradients $\nabla h_i(x^*)$ are linearly independent**
  - **The functions h_i are affine**

ENHANCED FRITZ JOHN CONDITIONS

If x^* is optimal, there exist $\mu_0 \geq 0$ and λ_i, not all 0, such that

$$\mu_0 \nabla f(x^*) + \sum_i \lambda_i \nabla h_i(x^*) = 0,$$

and a sequence $\{x^k\}$ with $x^k \to x^*$ and such that

$$f(x^k) < f(x^*) \text{ for all } k,$$

$$\lambda_i h_i(x^k) > 0 \text{ for all } i \text{ with } \lambda_i \neq 0 \text{ and all } k$$

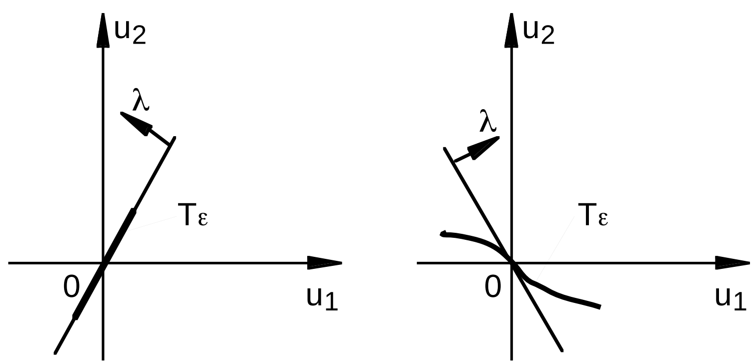
NOTE: If $\mu_0 > 0$, the λ_i are Lagrange multipliers with a sensitivity property (we call them **informative**)

PSEUDONORMALITY

- **Definition: A feasible point x^* is pseudonormal if one cannot find λ_i and a sequence $\{x^k\}$ with $x^k \to x^*$ such that**

  $$\textstyle\sum_i \lambda_i \nabla h_i(x^*) = 0, \qquad \sum_i \lambda_i h_i(x^k) > 0 \quad \text{for all } k$$

- **Pseudonormality at x^* guarantees that, if x^* is optimal, we can take $\mu_0 = 1$ in the F-J conditions (so there exists an informative Lagrange multiplier)**



Pseudonormal
h_i: Affine

Not Pseudonormal

Map an ε -ball around x^* onto the constraint space $T_\varepsilon = \{h(x) \mid \|x-x^*\| < \varepsilon\}$

INFORMATIVE LAGRANGE MULTIPLIERS

- **The Lagrange multipliers obtained from the enhanced Fritz John conditions have a special sensitivity property:**

  They indicate the constraints to violate in order to improve the cost

- **We call such multipliers informative**

- **Proposition: If there exists at least one Lagrange multiplier vector, there exists one that is informative**

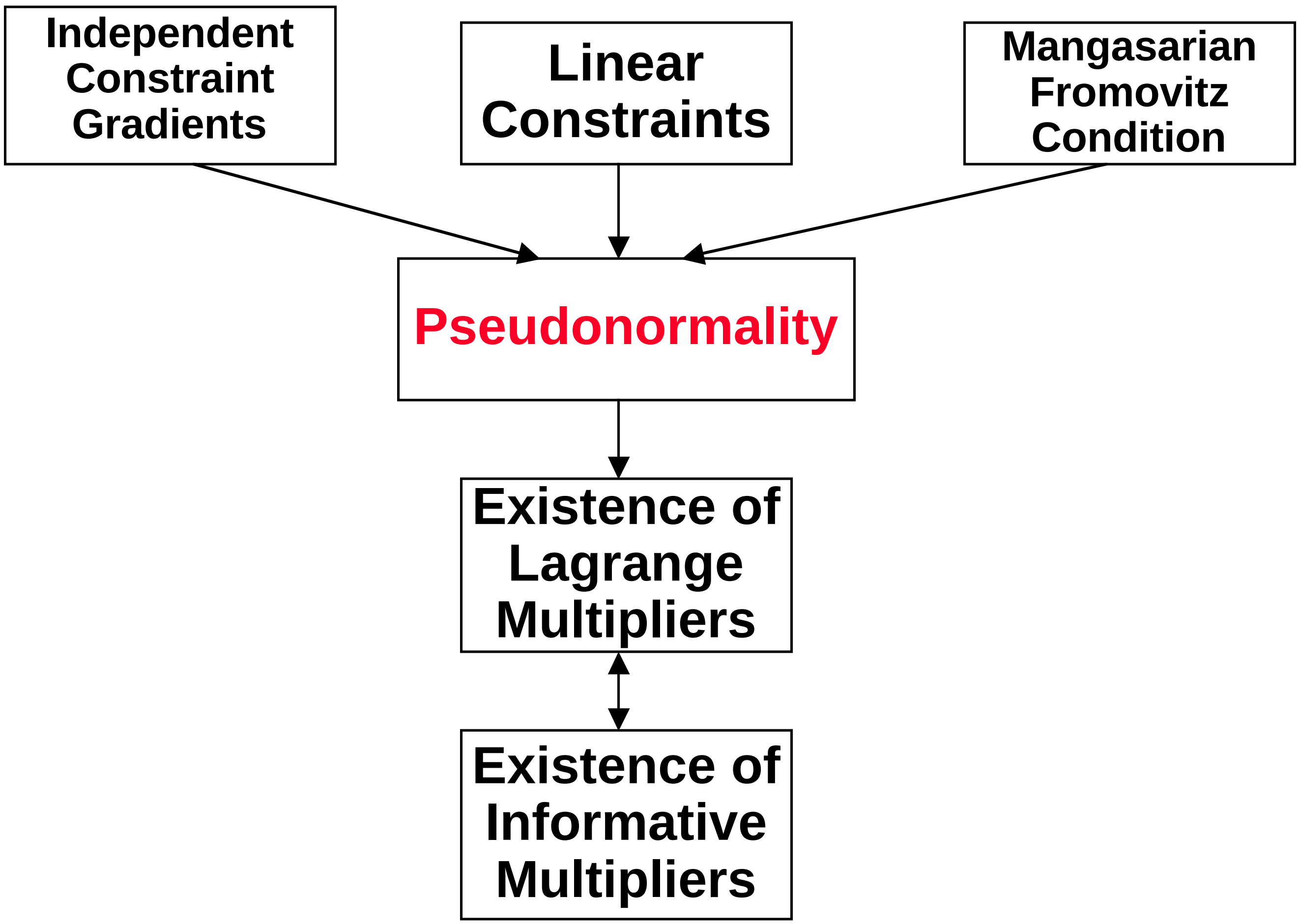
EXTENSIONS/CONNECTIONS TO NONSMOOTH ANALAYSIS

- **F-J conditions for an additional constraint $x \in X$**
- **The stationarity condition becomes**

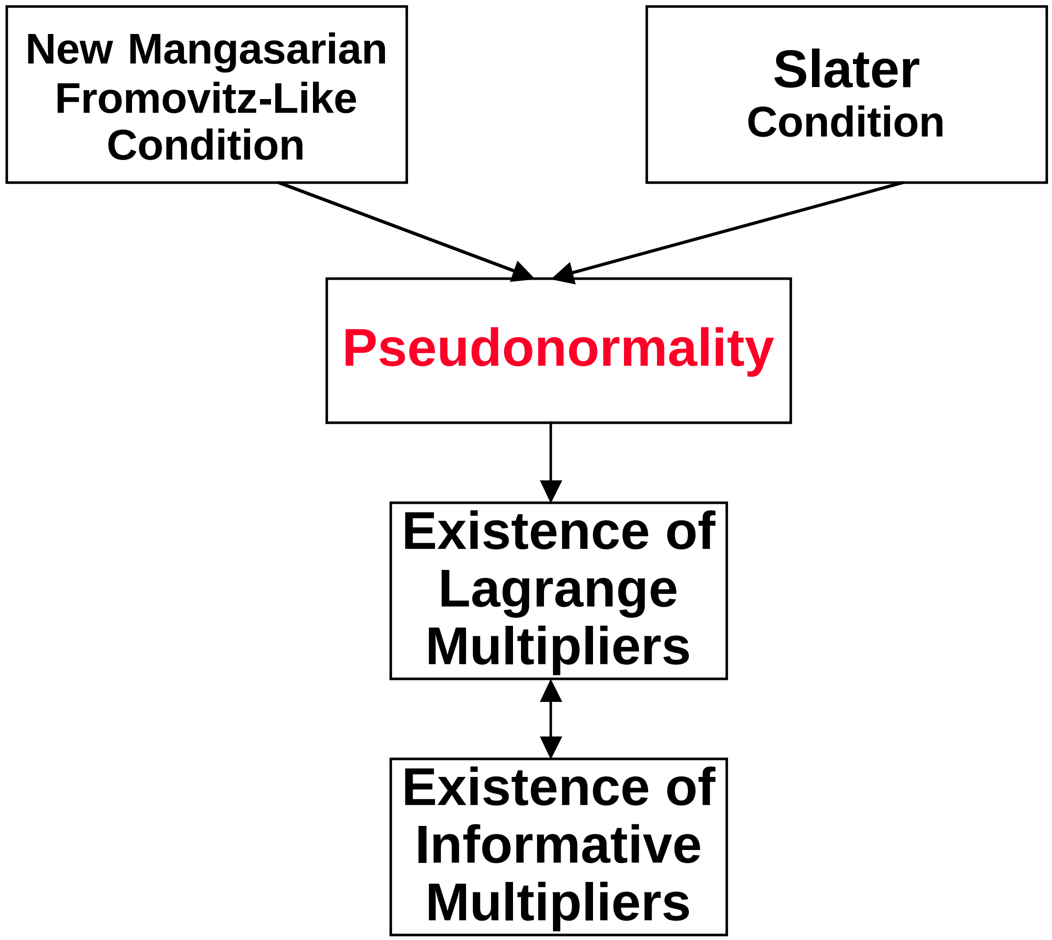
  $- \left(\nabla f(x^*) + \Sigma_I \lambda_i \nabla h_i(x^*) \right) \in$ (normal cone of X at x*)

- **X is called regular at x* if the normal cone is equal to the polar of its tangent cone at x* (example: X convex)**
- **If X is not regular at x*, the Lagrangian may have negative slope along some feasible directions**
- **Regularity is the fault line beyond which there is no satisfactory Lagrange multiplier theory**

THE STRUCTURE OF THE THEORY FOR $X = \Re^n$



THE STRUCTURE OF THE THEORY FOR X: REGULAR



EXTENSIONS

- Enhanced Fritz John conditions and pseudonormality for convex problems, when existence of a primal optimal solution is not assumed
- Connection of pseudonormality and exact penalty functions
- Connection of pseudonormality and the classical notion of quasiregularity