Convex Optimization Theory

Chapter 2

Exercises and Solutions

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CHAPTER 2: EXERCISES AND SOLUTIONS†

2.1

Let C be a nonempty convex subset of $\Re^n$, and let A be an $m \times n$ matrix with linearly independent columns. Show that a vector $x \in C$ is an extreme point of C if and only if Ax is an extreme point of the image AC. Show by example that if the columns of A are linearly dependent, then Ax can be an extreme point of AC, for some non-extreme point x of C.

Solution: Suppose that x is not an extreme point of C. Then $x = \alpha x_1 + (1-\alpha)x_2$ for some $x_1, x_2 \in C$ with $x_1 \neq x$ and $x_2 \neq x$, and a scalar $\alpha \in (0,1)$, so that $Ax = \alpha Ax_1 + (1-\alpha)Ax_2$. Since the columns of A are linearly independent, we have $Ay_1 = Ay_2$ if and only if $y_1 = y_2$. Therefore, $Ax_1 \neq Ax$ and $Ax_2 \neq Ax$, implying that Ax is a convex combination of two distinct points in AC, i.e., Ax is not an extreme point of AC.

Suppose now that Ax is not an extreme point of AC, so that $Ax = \alpha Ax_1 + (1-\alpha)Ax_2$ for some $x_1, x_2 \in C$ with $Ax_1 \neq Ax$ and $Ax_2 \neq Ax$, and a scalar $\alpha \in (0,1)$. Then, $A\big(x - \alpha x_1 - (1-\alpha)x_2\big) = 0$ and since the columns of A are linearly independent, it follows that $x = \alpha x_1 - (1-\alpha)x_2$. Furthermore, because $Ax_1 \neq Ax$ and $Ax_2 \neq Ax$, we must have $x_1 \neq x$ and $x_2 \neq x$, implying that x is not an extreme point of C.

As an example showing that if the columns of A are linearly dependent, then Ax can be an extreme point of AC, for some non-extreme point x of C, consider the 1×2 matrix $A = [1\ 0]$, whose columns are linearly dependent. The polyhedral set C given by

$$C = \big\{(x_1, x_2) \mid x_1 \geq 0,\ 0 \leq x_2 \leq 1\big\}$$

has two extreme points, (0,0) and (0,1). Its image $AC \subset \Re$ is given by

$$AC = \{x_1 \mid x_1 \geq 0\},$$

whose unique extreme point is $x_1 = 0$. The point $x = (0, 1/2) \in C$ is not an extreme point of C, while its image $Ax = 0$ is an extreme point of AC. Actually,

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† Many of the exercises and solutions given here were developed as part of my earlier convex optimization book [BNO03] (coauthored with Angelia Nedić and Asuman Ozdaglar), and are posted on the internet of that book's web site. The contribution of my coauthors in the development of these exercises and their solutions is gratefully acknowledged. Since some of the exercises and/or their solutions have been modified and also new exercises have been added, all errors are my sole responsibility.

all the points in C on the line segment connecting (0,0) and (0,1), except for (0,0) and (0,1), are non-extreme points of C that are mapped under A into the extreme point 0 of AC.

2.2

Show by example that the set of extreme points of a nonempty compact set need not be closed. *Hint*: Consider a line segment $C_1 = \big\{(x_1, x_2, x_3) \mid x_1 = 0, x_2 = 0, -1 \leq x_3 \leq 1\big\}$ and a circular disk $C_2 = \big\{(x_1, x_2, x_3) \mid (x_1 - 1)^2 + x_2^2 \leq 1,\ x_3 = 0\big\}$, and verify that the set $\mathrm{conv}(C_1 \cup C_2)$ is compact, while its set of extreme points is not closed.

Solution: For the sets C_1 and C_2 as given in this exercise, the set $C_1 \cup C_2$ is compact, and its convex hull is also compact by Prop. 1.2.2 of Chapter 1. The set of extreme points of $\mathrm{conv}(C_1 \cup C_2)$ is not closed, since it consists of the two end points of the line segment C_1, namely $(0, 0, -1)$ and $(0, 0, 1)$, and all the points $x = (x_1, x_2, x_3)$ such that

$$x \neq 0, \qquad (x_1 - 1)^2 + x_2^2 = 1, \qquad x_3 = 0.$$

2.3

Let C be a nonempty closed convex subset of $\Re^n$. Show that the following are equivalent.

(i) All boundary points of C are extreme points of C.

(ii) Every hyperplane that supports C at some point intersects C only at that point.

(iii) Every line intersects the boundary of C at no more than two points.

Solution: The result is clearly true if C consists of a single point, so assume that C consists of more than one point.

We first show that (i) implies (ii). Assume that all boundary points of C are extreme points. If there is a hyperplane that supports C and intersects C at two distinct points, the entire line segment connecting the two points would lie on the boundary of C, but the midpoint of this line segment would not be an extreme point - a contradiction.

Next we show that (ii) implies (iii). Assume that every hyperplane that supports C at some point intersects C only at that point. Suppose that there is a line that intersects the boundary of C at three distinct boundary points x_1, x_2, x_3, with x_2 being the midpoint. Consider a hyperplane H that supports C at x_2, i.e., a vector $a \neq 0$ such that

$$a'x \geq a'x_2, \qquad \forall\ x \in C.$$

Then since by the hypothesis, H intersects C only at x_2, we must have $a'x_1 > a'x_2$ and $a'x_3 > a'x_2$, which is a contradiction since x_2 lies strictly between x_1 and x_3.

Finally, we show that (iii) implies (i). Assume that every line intersects the boundary of C at no more than two points. If there is a boundary point x_2 that is not extreme and therefore lies strictly between two points $x_1, x_3 \in C$, then either x_1 or x_3 must be an interior point, for otherwise the line that passes through x_1, x_2, x_3 would contain more than two boundary points. Thus, by the Line Segment Principle (Prop. 1.3.1), every point that lies strictly between x_1 and x_3, including x_2, is an interior point of C. This contradicts the hypothesis that x_2 is a boundary point of C.

2.4 (Matrix Inequalities)

Let A be a symmetric $n \times n$ matrix with components denoted a_{ij} and eigenvalues denoted $\lambda_1, \ldots, \lambda_n$, and let Λ_A be the set of all vectors of $\Re^n$ obtained by permutations of these eigenvalues.

(a) Let C be a convex set that contains Λ_A, and let $f : C \mapsto \Re^n$ be a convex function. Show that for any orthonormal set of vectors $v_1, \ldots, v_n$ in $\Re^n$, we have

$$f(v_1'Av_1, \ldots, v_n'Av_n) \le \max_{(\xi_1,\ldots,\xi_n)\in\Lambda_A} f(\xi_1, \ldots, \xi_n).$$

Hint: Let S be the doubly stochastic matrix with components $s_{ij} = (v_i'u_j)^2$, where $u_1, \ldots, u_n$ are orthonormal eigenvectors corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$. Show that $v = S\lambda$, where

$$v = (v_1'Av_1, \ldots, v_n'Av_n), \qquad \lambda = (\lambda_1, \ldots, \lambda_n),$$

and use the Birkhoff-von Neumann Theorem.

(b) Let A be positive semidefinite. Show that for any orthonormal set of vectors $v_1, \ldots, v_n$ in $\Re^n$, we have

$$\det A = \lambda_1 \cdots \lambda_n \le v_1'Av_1 \cdots v_n'Av_n.$$

Furthermore, the inequality is sharp in the sense that it is satisfied as an equality for some orthonormal set of vectors. *Hint*: Use part (a) with $f(x_1, \ldots, x_n) = -(x_1 \cdots x_n)^{1/2}$, and C equal to the nonnegative orthant.

(c) *(Hadamard's Determinant Inequality)* We have

$$(\det A)^2 \le (a_{11}^2 + \cdots + a_{n1}^2) \cdots (a_{1n}^2 + \cdots + a_{nn}^2).$$

Furthermore, if in addition A is positive semidefinite, we have

$$\det A \le a_{11} \cdots a_{nn}.$$

Solution: (a) Let $u_1, \ldots, u_n$ be orthonormal eigenvectors corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$. The orthogonality of $u_1, \ldots, u_n$ implies that

$$v_i = (v_i'u_1)u_1 + \cdots + (v_i'u_n)u_n, \qquad i = 1, \ldots, n.$$

Using this relation, it is straightforward to verify that

$$v = S\lambda,$$

where

$$v = (v_1'Av_1, \ldots, v_n'Av_n), \qquad \lambda = (\lambda_1, \ldots, \lambda_n),$$

and S is the $n \times n$ matrix with components $s_{ij} = (v_i'u_j)^2$ for all i and j. We now note that S is a doubly stochastic matrix. The reason is that we have for each i,

$$\|v_i\|^2 = \big\|(v_i'u_1)u_1 + \cdots + (v_i'u_n)u_n\big\|^2,$$

so that by using the orthonormality of $u_1, \ldots, u_n$, we have

$$\|v_i\|^2 = (v_i'u_1)^2 + \cdots + (v_i'u_n)^2.$$

This implies that S is doubly stochastic, since $\|v_i\| = 1$ by assumption, and the ith row of the matrix S is $\big((v_i'u_1)^2, \ldots, (v_i'u_n)^2\big)$.

The Birkhoff-von Neumann Theorem asserts that S can be expressed as a convex combination of permutation matrices, i.e., there exist $\mu_j \geq 0$, $j = 1, \ldots, m$, with $\sum_{j=1}^m \mu_j = 1$, and such that

$$S = \mu_1 P_1 + \cdots + \mu_m P_m,$$

where $P_1, \ldots, P_m$ are permutation matrices. Hence,

$$v = S\lambda = \mu_1(P_1\lambda) + \cdots + \mu_m(P_m\lambda).$$

Since the vectors $P_j\lambda$, $j = 1, \ldots, m$, belong to Λ_A, they also belong to C. Since v is a convex combination of $P_j\lambda$, $j = 1, \ldots, m$, it follows that $v \in C$. Thus, using the convexity of f, we have

$$f(v) \leq \mu_1 f(P_1\lambda) + \cdots + \mu_m f(P_m\lambda) \leq \max_{(\xi_1, \ldots, \xi_n) \in \Lambda_A} f(\xi_1, \ldots, \xi_n).$$

(b) The inequality follows from part (a) and the hint. The inequality is satisfied as an equality if the vectors $v_1, \ldots, v_n$ are normalized eigenvectors corresponding to $\lambda_1, \ldots, \lambda_n$.

(c) Let $B = A'A$. We apply part (b) to B with the orthonormal vectors being the unit vectors $e_1, \ldots, e_n$ of $\Re^n$. We obtain

$$\det B \leq e_1'Be_1 \cdots e_n'Be_n = (a_{11}^2 + \cdots + a_{n1}^2) \cdots (a_{1n}^2 + \cdots + a_{nn}^2),$$

where the last equality can be verified by straightforward calculation. Since $\det B = (\det A)^2$, the desired inequality follows.

If A positive semidefinite, we apply part (b) to A with the orthonormal vectors being the unit vectors $e_1, \ldots, e_n$ of $\Re^n$, to obtain

$$\det A \leq a_{11} \cdots a_{nn}.$$

2.5 (Faces)

Let P be a polyhedral set. For any hyperplane H that passes through a boundary point of P and contains P in one of its halfspaces, we say that the set $F = P \cap H$ is a *face* of P. Show the following:

(a) Each face is a polyhedral set.

(b) Each extreme point of P, viewed as a singleton set, is a face.

(c) If P is not an affine set, there is a face of P whose dimension is $\dim(P) - 1$.

(d) The number of distinct faces of P is finite.

Solution: (a) Let P be a polyhedral set in $\Re^n$, and let $F = P \cap H$ be a face of P, where H is a hyperplane passing through some boundary point $\overline{x}$ of P and containing P in one of its halfspaces. Then H is given by $H = \{x \mid a'x = a'\overline{x}\}$ for some nonzero vector $a \in \Re^n$. By replacing $a'x = a'\overline{x}$ with two inequalities $a'x \le a'\overline{x}$ and $-a'x \le -a'\overline{x}$, we see that H is a polyhedral set in $\Re^n$. Since the intersection of two nondisjoint polyhedral sets is a polyhedral set, the set $F = P \cap H$ is polyhedral.

(b) Let P be given by

$$P = \big\{x \mid a_j'x \le b_j,\ j = 1, \ldots, r\big\},$$

for some vectors $a_j \in \Re^n$ and scalars b_j. Let v be an extreme point of P, and without loss of generality assume that the first n inequalities define v, i.e., the first n of the vectors a_j are linearly independent and such that

$$a_j'v = b_j, \qquad \forall\ j = 1, \ldots, n$$

[cf. Prop. 2.1.4(a)]. Define the vector $a \in \Re^n$, the scalar b, and the hyperplane H as follows

$$a = \frac{1}{n}\sum_{j=1}^{n} a_j, \qquad b = \frac{1}{n}\sum_{j=1}^{n} b_j, \qquad H = \big\{x \mid a'x = b\big\}.$$

Then, we have

$$a'v = b,$$

so that H passes through v. Moreover, for every $x \in P$, we have $a_j'x \le b_j$ for all j, implying that $a'x \le b$ for all $x \in P$. Thus, H contains P in one of its halfspaces.

We will next prove that $P \cap H = \{v\}$. We start by showing that for every $\overline{v} \in P \cap H$, we must have

$$a_j'\overline{v} = b_j, \qquad \forall\ j = 1, \ldots, n. \tag{2.1}$$

To arrive at a contradiction, assume that $a_j'\overline{v} < b_j$ for some $\overline{v} \in P \cap H$ and $j \in \{1, \ldots, n\}$. Without loss of generality, we can assume that the strict inequality holds for $j = 1$, so that

$$a_1'\overline{v} < b_1, \qquad a_j'\overline{v} \le b_j, \qquad \forall\ j = 2, \ldots, n.$$

By multiplying each of the above inequalities with $1/n$ and by summing the obtained inequalities, we obtain

$$\frac{1}{n}\sum_{j=1}^{n} a_j'\overline{v} < \frac{1}{n}\sum_{j=1}^{n} b_j,$$

implying that $a'\overline{v} < b$, which contradicts the fact that $\overline{v} \in H$. Hence, Eq. (2.1) holds, and since the vectors $a_1, \ldots, a_n$ are linearly independent, it follows that $v = \overline{v}$, showing that $P \cap H = \{v\}$.

As discussed in Section 2.1, every extreme point of P is a relative boundary point of P. Since every relative boundary point of P is also a boundary point of P, it follows that every extreme point of P is a boundary point of P. Thus, v is a boundary point of P, and as shown earlier, H passes through v and contains P in one of its halfspaces. By definition, it follows that $P \cap H = \{v\}$ is a face of P.

(c) Since P is not an affine set, it cannot consist of a single point, so we must have $\dim(P) > 0$. Let P be given by

$$P = \bigl\{x \mid a_j'x \le b_j,\ j = 1, \ldots, r\bigr\},$$

for some vectors $a_j \in \Re^n$ and scalars b_j. Also, let A be the matrix with rows a_j' and b be the vector with components b_j, so that

$$P = \{x \mid Ax \le b\}.$$

An inequality $a_j'x \le b_j$ of the system $Ax \le b$ is *redundant* if it is implied by the remaining inequalities in the system. If the system $Ax \le b$ has no redundant inequalities, we say that the system is *nonredundant*. An inequality $a_j'x \le b_j$ of the system $Ax \le b$ is an *implicit equality* if $a_j'x = b_j$ for all x satisfying $Ax \le b$.

By removing the redundant inequalities if necessary, we may assume that the system $Ax \le b$ defining P is nonredundant. Since P is not an affine set, there exists an inequality $a_{j_0}'x \le b_{j_0}$ that is not an implicit equality of the system $Ax \le b$. Consider the set

$$F = \bigl\{x \in P \mid a_{j_0}'x = b_{j_0}\bigr\}.$$

Note that $F \ne \emptyset$, since otherwise $a_{j_0}'x \le b_{j_0}$ would be a redundant inequality of the system $Ax \le b$, contradicting our earlier assumption that the system is nonredundant. Note also that every point of F is a boundary point of P. Thus, F is the intersection of P and the hyperplane $\bigl\{x \mid a_{j_0}'x = b_{j_0}\bigr\}$ that passes through a boundary point of P and contains P in one of its halfspaces, i.e., F is a face of P. Since $a_{j_0}'x \le b_{j_0}$ is not an implicit equality of the system $Ax \le b$, the dimension of F is $\dim(P) - 1$.

(d) Let P be a polyhedral set given by

$$P = \bigl\{x \mid a_j'x \le b_j,\ j = 1, \ldots, r\bigr\},$$

with $a_j \in \Re^n$ and $b_j \in \Re$, or equivalently

$$P = \{x \mid Ax \le b\},$$

where A is an $r \times n$ matrix and $b \in \Re^r$. We will show that F is a face of P if and only if F is nonempty and

$$F = \big\{ x \in P \mid a_j' x = b_j,\ j \in J \big\},$$

where $J \subset \{1, \ldots, r\}$. From this it will follow that the number of distinct faces of P is finite.

By removing the redundant inequalities if necessary, we may assume that the system $Ax \le b$ defining P is nonredundant. Let F be a face of P, so that $F = P \cap H$, where H is a hyperplane that passes through a boundary point of P and contains P in one of its halfspaces. Let $H = \big\{ x \mid\ c'x = c\overline{x} \big\}$ for a nonzero vector $c \in \Re^n$ and a boundary point $\overline{x}$ of P, so that

$$F = \big\{ x \in P \mid c'x = c\overline{x} \big\}$$

and

$$c'x \le c\overline{x}, \qquad \forall\ x \in P.$$

These relations imply that the set of points x such that $Ax \le b$ and $c'x \le c\overline{x}$ coincides with P, and since the system $Ax \le b$ is nonredundant, it follows that $c'x \le c\overline{x}$ is a redundant inequality of the system $Ax \le b$ and $c'x \le c\overline{x}$. Therefore, the inequality $c'x \le c\overline{x}$ is implied by the inequalities of $Ax \le b$, so that there exists some $\mu \in \Re^r$ with $\mu \ge 0$ such that

$$\sum_{j=1}^{r} \mu_j a_j = c, \qquad \sum_{j=1}^{r} \mu_j b_j = c'\overline{x}.$$

Let $J = \{ j \mid \mu_j > 0 \}$. Then, for every $x \in P$, we have

$$c'x = c\overline{x} \quad \Longleftrightarrow \quad \sum_{j \in J} \mu_j a_j' x = \sum_{j \in J} \mu_j b_j \quad \Longleftrightarrow \quad a_j' x = b_j,\ j \in J, \tag{2.2}$$

implying that

$$F = \big\{ x \in P \mid a_j' x = b_j, j \in J \big\}.$$

Conversely, let F be a nonempty set given by

$$F = \big\{ x \in P \mid a_j' x = b_j, j \in J \big\},$$

for some $J \subset \{1, \ldots, r\}$. Define

$$c = \sum_{j \in J} a_j, \qquad \beta = \sum_{j \in J} b_j.$$

Then, we have

$$\big\{ x \in P \mid a_j' x = b_j, j \in J \big\} = \big\{ x \in P \mid c'x = \beta \big\},$$

[cf. Eq. (2.2) where $\mu_j = 1$ for all $j \in J$]. Let $H = \big\{ x \mid c'x = \beta \big\}$, so that in view of the preceding relation, we have that $F = P \cap H$. Since every point of F is a boundary point of P, it follows that H passes through a boundary point of P. Furthermore, for every $x \in P$, we have $a_j' x \le b_j$ for all $j \in J$, implying that $c'x \le \beta$ for every $x \in P$. Thus, H contains P in one of its halfspaces. Hence, F is a face.

2.6 (Isomorphic Polyhedral Sets)

Let P and Q be polyhedral sets in $\Re^n$ and $\Re^m$, respectively. We say that P and Q are *isomorphic* if there exist affine functions $f : P \mapsto Q$ and $g : Q \mapsto P$ such that

$$x = g\big(f(x)\big), \quad \forall\ x \in P, \qquad\quad y = f\big(g(y)\big), \quad \forall\ y \in Q.$$

(a) Show that if P and Q are isomorphic, then their extreme points are in one-to-one correspondence.

(b) Let A be an $r \times n$ matrix and b be a vector in $\Re^r$, and let

$$P = \{x \in \Re^n \mid Ax \le b,\ x \ge 0\},$$

$$Q = \big\{(x, z) \in \Re^{n+r} \mid Ax + z = b,\ x \ge 0,\ z \ge 0\big\}.$$

Show that P and Q are isomorphic.

Solution: (a) Let P and Q be isomorhic polyhedral sets, and let $f : P \mapsto Q$ and $g : Q \mapsto P$ be affine functions such that

$$x = g\big(f(x)\big), \quad \forall\ x \in P, \qquad\quad y = f\big(g(y)\big), \quad \forall\ y \in Q.$$

Assume that x^* is an extreme point of P and let $y^* = f(x^*)$. We will show that y^* is an extreme point of Q. Since x^* is an extreme point of P, by Exercise 2.5(b), it is also a face of P, and therefore, there exists a vector $c \in \Re^n$ such that

$$c'x < c'x^*, \qquad \forall\ x \in P,\ x \ne x^*.$$

For any $y \in Q$ with $y \ne y^*$, we have

$$f\big(g(y)\big) = y \ne y^* = f(x^*),$$

implying that

$$g(y) \ne g(y^*) = x^*, \qquad \text{with } \ g(y) \in P.$$

Hence,

$$c'g(y) < c'g(y^*), \qquad \forall\ y \in Q,\ y \ne y^*.$$

Let the affine function g be given by $g(y) = By + d$ for some $n \times m$ matrix B and vector $d \in \Re^n$. Then, we have

$$c'(By + d) < c'(By^* + d), \qquad \forall\ y \in Q,\ y \ne y^*,$$

implying that

$$(B'c)'y < (B'c)'y^*, \qquad \forall\ y \in Q,\ y \ne y^*.$$

If y^* were not an extreme point of Q, then we would have $y^* = \alpha y_1 + (1 - \alpha)y_2$ for some distinct points $y_1, y_2 \in Q$, $y_1 \ne y^*$, $y_2 \ne y^*$, and $\alpha \in (0, 1)$, so that

$$(B'c)'y^* = \alpha(B'c)'y_1 + (1 - \alpha)(B'c)'y_2 < (B'c)'y^*,$$

which is a contradiction. Hence, y^* is an extreme point of Q.

Conversely, if y^* is an extreme point of Q, then by using a symmetrical argument, we can show that x^* is an extreme point of P.

(b) For the sets

$$P = \{x \in \Re^n \mid Ax \le b,\ x \ge 0\},$$

$$Q = \big\{(x,z) \in \Re^{n+r} \mid Ax + z = b,\ x \ge 0,\ z \ge 0\big\},$$

let f and g be given by

$$f(x) = (x, b - Ax), \qquad \forall\ x \in P,$$

$$g(x,z) = x, \qquad \forall\ (x,z) \in Q.$$

Evidently, f and g are affine functions. Furthermore, clearly

$$f(x) \in Q, \qquad g\big(f(x)\big) = x, \qquad \forall\ x \in P,$$

$$g(x,z) \in P, \qquad f\big(g(x,z)\big) = x, \qquad \forall\ (x,z) \in Q.$$

Hence, P and Q are isomorphic.

2.7 (Cone Decomposition Theorem)

Let C be a nonempty closed convex cone in $\Re^n$ and let x be a vector in $\Re^n$. Show that:

(a) $\hat{x}$ is the projection of x on C if and only if

$$\hat{x} \in C, \qquad (x - \hat{x})'\hat{x} = 0, \qquad x - \hat{x} \in C^*.$$

(b) The following two statements are equivalent:

  (i) x_1 and x_2 are the projections of x on C and C^*, respectively.

  (ii) $x = x_1 + x_2$ with $x_1 \in C$, $x_2 \in C^*$, and $x_1'x_2 = 0$.

Solution: (a) Let $\hat{x}$ be the projection of x on C, which exists and is unique since C is closed and convex. By the Projection Theorem (Prop. 1.1.9), we have

$$(x - \hat{x})'(y - \hat{x}) \le 0, \qquad \forall\ y \in C.$$

Since C is a cone, we have $(1/2)\hat{x} \in C$ and $2\hat{x} \in C$, and by taking $y = (1/2)\hat{x}$ and $y = 2\hat{x}$ in the preceding relation, it follows that

$$(x - \hat{x})'\hat{x} = 0.$$

By combining the preceding two relations, we obtain

$$(x - \hat{x})'y \le 0, \qquad \forall\ y \in C,$$

implying that $x - \hat{x} \in C^*$.

Conversely, if $\hat{x} \in C$, $(x - \hat{x})'\hat{x} = 0$, and $x - \hat{x} \in C^*$, then it follows that

$$(x - \hat{x})'(y - \hat{x}) \le 0, \qquad \forall\, y \in C,$$

and by the Projection Theorem, $\hat{x}$ is the projection of x on C.

(b) Suppose that property (i) holds, i.e., x_1 and x_2 are the projections of x on C and C^*, respectively. Then, by part (a), we have

$$x_1 \in C, \qquad (x - x_1)'x_1 = 0, \qquad x - x_1 \in C^*.$$

Let $y = x - x_1$, so that the preceding relation can equivalently be written as

$$x - y \in C = (C^*)^*, \qquad y'(x - y) = 0, \qquad y \in C^*.$$

By using part (a), we conclude that y is the projection of x on C^*. Since by the Projection Theorem, the projection of a vector on a closed convex set is unique, it follows that $y = x_2$. Thus, we have $x = x_1 + x_2$ and in view of the preceding two relations, we also have $x_1 \in C$, $x_2 \in C^*$, and $x_1'x_2 = 0$. Hence, property (ii) holds.

Conversely, suppose that property (ii) holds, i.e., $x = x_1 + x_2$ with $x_1 \in C$, $x_2 \in C^*$, and $x_1'x_2 = 0$. Then, evidently the relations

$$x_1 \in C, \qquad (x - x_1)'x_1 = 0, \qquad x - x_1 \in C^*,$$

$$x_2 \in C^*, \qquad (x - x_2)'x_2 = 0, \qquad x - x_2 \in C$$

are satisfied, so that by part (a), x_1 and x_2 are the projections of x on C and C^*, respectively. Hence, property (i) holds.

2.8 (Polar Cone Operations)

Show the following:

(a) For any nonempty cones $C_i \subset \Re^{n_i}$, $i = 1, \ldots, m$, we have

$$(C_1 \times \cdots \times C_m)^* = C_1^* \times \cdots \times C_m^*.$$

(b) For any collection of nonempty cones $\{C_i \mid i \in I\}$, we have

$$\big(\cup_{i \in I} C_i\big)^* = \cap_{i \in I} C_i^*.$$

(c) For any two nonempty cones C_1 and C_2, we have

$$(C_1 + C_2)^* = C_1^* \cap C_2^*.$$

(d) For any two nonempty closed convex cones C_1 and C_2, we have

$$(C_1 \cap C_2)^* = \mathrm{cl}(C_1^* + C_2^*).$$

Furthermore, if $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) \neq \emptyset$, then the cone $C_1^* + C_2^*$ is closed and the closure operation in the preceding relation can be omitted.

(e) Consider the following cones in $\Re^3$

$$C_1 = \big\{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \leq x_3^2,\ x_3 \leq 0\big\},$$

$$C_2 = \big\{(x_1, x_2, x_3) \mid x_2 = -x_3\big\}.$$

Verify that $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) = \emptyset$, $(1, 1, 1) \in (C_1 \cap C_2)^*$, and $(1, 1, 1) \notin C_1^* + C_2^*$, thus showing that the closure operation in the relation of part (c) may not be omitted when $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) = \emptyset$.

Solution: (a) It suffices to consider the case where $m = 2$. Let $(y_1, y_2) \in (C_1 \times C_2)^*$. Then, we have $(y_1, y_2)'(x_1, x_2) \leq 0$ for all $(x_1, x_2) \in C_1 \times C_2$, or equivalently

$$y_1' x_1 + y_2' x_2 \leq 0, \qquad \forall\ x_1 \in C_1, \quad \forall\ x_2 \in C_2.$$

Since C_2 is a cone, 0 belongs to its closure, so by letting $x_2 \to 0$ in the preceding relation, we obtain $y_1' x_1 \leq 0$ for all $x_1 \in C_1$, showing that $y_1 \in C_1^*$. Similarly, we obtain $y_2 \in C_2^*$, and therefore $(y_1, y_2) \in C_1^* \times C_2^*$, implying that $(C_1 \times C_2)^* \subset C_1^* \times C_2^*$.

Conversely, let $y_1 \in C_1^*$ and $y_2 \in C_2^*$. Then, we have

$$(y_1, y_2)'(x_1, x_2) = y_1' x_1 + y_2' x_2 \leq 0, \qquad \forall\ x_1 \in C_1, \quad \forall\ x_2 \in C_2,$$

implying that $(y_1, y_2) \in (C_1 \times C_2)^*$, and showing that $C_1^* \times C_2^* \subset (C_1 \times C_2)^*$.

(b) A vector y belongs to the polar cone of $\cup_{i \in I} C_i$ if and only if $y'x \leq 0$ for all $x \in C_i$ and all $i \in I$, which is equivalent to having $y \in C_i^*$ for every $i \in I$. Hence, y belongs to $\big(\cup_{i \in I} C_i\big)^*$ if and only if y belongs to $\cap_{i \in I} C_i^*$.

(c) Let $y \in (C_1 + C_2)^*$, so that

$$y'(x_1 + x_2) \leq 0, \qquad \forall\ x_1 \in C_1, \qquad \forall\ x_2 \in C_2. \tag{2.3}$$

Since the zero vector is in the closures of C_1 and C_2, by letting $x_2 \to 0$ with $x_2 \in C_2$ in Eq. (2.3), we obtain

$$y' x_1 \leq 0, \qquad \forall\ x_1 \in C_1,$$

and similarly, by letting $x_1 \to 0$ with $x_1 \in C_1$ in Eq. (2.3), we obtain

$$y' x_2 \leq 0, \qquad \forall\ x_2 \in C_2.$$

Thus, $y \in C_1^* \cap C_2^*$, showing that $(C_1 + C_2)^* \subset C_1^* \cap C_2^*$.

Conversely, let $y \in C_1^* \cap C_2^*$. Then, we have

$$y' x_1 \leq 0, \qquad \forall\ x_1 \in C_1,$$

$$y' x_2 \leq 0, \qquad \forall\ x_2 \in C_2,$$

implying that

$$y'(x_1 + x_2) \le 0, \qquad \forall\, x_1 \in C_1, \quad \forall\, x_2 \in C_2.$$

Hence $y \in (C_1 + C_2)^*$, showing that $C_1^* \cap C_2^* \subset (C_1 + C_2)^*$.

(d) Since C_1 and C_2 are closed convex cones, by the Polar Cone Theorem (Prop. 2.2.1) and by part (b), it follows that

$$C_1 \cap C_2 = (C_1^*)^* \cap (C_2^*)^* = (C_1^* + C_2^*)^*.$$

By taking the polars and by using the Polar Cone Theorem, we obtain

$$(C_1 \cap C_2)^* = \big((C_1^* + C_2^*)^*\big)^* = \mathrm{cl}\big(\mathrm{conv}(C_1^* + C_2^*)\big).$$

The cone $C_1^* + C_2^*$ is convex, so that

$$(C_1 \cap C_2)^* = \mathrm{cl}(C_1^* + C_2^*).$$

Suppose now that $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) \neq \emptyset$. We will show that $C_1^* + C_2^*$ is closed by using Prop. 1.4.14. According to this proposition, if for any nonempty closed convex sets $\overline{C}_1$ and $\overline{C}_2$ in $\Re^n$, the equality $y_1 + y_2 = 0$ with $y_1 \in R_{\overline{C}_1}$ and $y_2 \in R_{\overline{C}_2}$ implies that y_1 and y_2 belong to the lineality spaces of $\overline{C}_1$ and $\overline{C}_2$, respectively, then the vector sum $\overline{C}_1 + \overline{C}_2$ is closed.

Let $y_1 + y_2 = 0$ with $y_1 \in R_{C_1^*}$ and $y_2 \in R_{C_2^*}$. Because C_1^* and C_2^* are closed convex cones, we have $R_{C_1^*} = C_1^*$ and $R_{C_2^*} = C_2^*$, so that $y_1 \in C_1^*$ and $y_2 \in C_2^*$. The lineality space of a cone is the set of vectors y such that y and $-y$ belong to the cone, so that in view of the preceding discussion, to show that $C_1^* + C_2^*$ is closed, it suffices to prove that $-y_1 \in C_1^*$ and $-y_2 \in C_2^*$.

Since $y_1 = -y_2$ and $y_1 \in C_1^*$, it follows that

$$y_2'x \ge 0, \qquad \forall\, x \in C_1, \tag{2.4}$$

and because $y_2 \in C_2^*$, we have

$$y_2'x \le 0, \qquad \forall\, x \in C_2,$$

which combined with the preceding relation yields

$$y_2'x = 0, \qquad \forall\, x \in C_1 \cap C_2. \tag{2.5}$$

In view of the fact $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) \neq \emptyset$, and Eqs. (2.4) and (2.5), it follows that the linear function $y_2'x$ attains its minimum over the convex set C_1 at a point in the relative interior of C_1, implying that $y_2'x = 0$ for all $x \in C_1$ (cf. Prop. 1.3.4). Therefore, $y_2 \in C_1^*$ and since $y_2 = -y_1$, we have $-y_1 \in C_1^*$. By exchanging the roles of y_1 and y_2 in the preceding analysis, we similarly show that $-y_2 \in C_2^*$, completing the proof.

(e) By drawing the cones C_1 and C_2, it can be seen that $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) = \emptyset$ and

$$C_1 \cap C_2 = \big\{(x_1, x_2, x_3) \mid x_1 = 0,\ x_2 = -x_3,\ x_3 \le 0\big\},$$

$$C_1^* = \big\{(y_1, y_2, y_3) \mid y_1^2 + y_2^2 \le y_3^2,\ y_3 \ge 0\big\},$$

$$C_2^* = \big\{(z_1, z_2, z_3) \mid z_1 = 0,\ z_2 = z_3\big\}.$$

Clearly, $x_1 + x_2 + x_3 = 0$ for all $x \in C_1 \cap C_2$, implying that $(1, 1, 1) \in (C_1 \cap C_2)^*$. Suppose that $(1, 1, 1) \in C_1^* + C_2^*$, so that $(1, 1, 1) = (y_1, y_2, y_3) + (z_1, z_2, z_3)$ for some $(y_1, y_2, y_3) \in C_1^*$ and $(z_1, z_2, z_3) \in C_2^*$, implying that $y_1 = 1$, $y_2 = 1 - z_2$, $y_3 = 1 - z_2$ for some $z_2 \in \Re$. However, this point does not belong to C_1^*, which is a contradiction. Therefore, $(1, 1, 1)$ is not in $C_1^* + C_2^*$. Hence, when $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) = \emptyset$, the relation

$$(C_1 \cap C_2)^* = C_1^* + C_2^*$$

may fail.

2.9 (Linear Transformations and Polar Cones)

Let C be a nonempty cone in $\Re^n$, K be a nonempty closed convex cone in $\Re^m$, and A be a linear transformation from $\Re^n$ to $\Re^m$. Show that

$$(AC)^* = (A')^{-1} \cdot C^*, \qquad \big(A^{-1} \cdot K\big)^* = \mathrm{cl}(A'K^*).$$

Show also that if $\mathrm{ri}(K) \cap R(A) \ne \emptyset$, then the cone $A'K^*$ is closed and $(A')^{-1}$ and the closure operation in the above relation can be omitted.

Solution: We have $y \in (AC)^*$ if and only if $y'Ax \le 0$ for all $x \in C$, which is equivalent to $(A'y)'x \le 0$ for all $x \in C$. This is in turn equivalent to $A'y \in C^*$. Hence, $y \in (AC)^*$ if and only if $y \in (A')^{-1} \cdot C^*$, showing that

$$(AC)^* = (A')^{-1} \cdot C^*. \tag{2.6}$$

We next show that for a closed convex cone $K \subset \Re^m$, we have

$$\big(A^{-1} \cdot K\big)^* = \mathrm{cl}(A'K^*).$$

Let $y \in \big(A^{-1} \cdot K\big)^*$ and to arrive at a contradiction, assume that $y \notin \mathrm{cl}(A'K^*)$. By the Strict Separation Theorem (Prop. 1.5.3), the closed convex cone $\mathrm{cl}(A'K^*)$ and the vector y can be strictly separated, i.e., there exist a vector $a \in \Re^n$ and a scalar b such that

$$a'x < b < a'y, \qquad \forall\ x \in \mathrm{cl}(A'K^*).$$

If $a'x > 0$ for some $x \in \mathrm{cl}(A'K^*)$, then since $\mathrm{cl}(A'K^*)$ is a cone, we would have $\lambda x \in \mathrm{cl}(A'K^*)$ for all $\lambda > 0$, implying that $a'(\lambda x) \to \infty$ when $\lambda \to \infty$, which contradicts the preceding relation. Thus, we must have $a'x \leq 0$ for all $x \in \mathrm{cl}(A'K^*)$, and since $0 \in \mathrm{cl}(A'K^*)$, it follows that

$$\sup_{x \in \mathrm{cl}(A'K^*)} a'x = 0 \leq b < a'y. \tag{2.7}$$

Therefore, $a \in \big(\mathrm{cl}(A'K^*)\big)^*$, and since $\big(\mathrm{cl}(A'K^*)\big)^* \subset (A'K^*)^*$, it follows that $a \in (A'K^*)^*$. In view of Eq. (2.6) and the Polar Cone Theorem (Prop. 2.2.1), we have

$$(A'K^*)^* = A^{-1}(K^*)^* = A^{-1} \cdot K,$$

implying that $a \in A^{-1} \cdot K$. Because $y \in \big(A^{-1} \cdot K\big)^*$, it follows that $y'a \leq 0$, contradicting Eq. (2.7). Hence, we must have $y \in \mathrm{cl}(A'K^*)$, showing that

$$\big(A^{-1} \cdot K\big)^* \subset \mathrm{cl}(A'K^*).$$

To show the reverse inclusion, let $y \in A'K^*$ and assume, to arrive at a contradiction, that $y \notin (A^{-1} \cdot K)^*$. By the Strict Separation Theorem (Prop. 1.5.3), the closed convex cone $(A^{-1} \cdot K)^*$ and the vector y can be strictly separated, i.e., there exist a vector $\overline{a} \in \Re^n$ and a scalar $\overline{b}$ such that

$$\overline{a}'x < \overline{b} < \overline{a}'y, \qquad \forall\ x \in (A^{-1} \cdot K)^*.$$

Similar to the preceding analysis, since $(A^{-1} \cdot K)^*$ is a cone, it can be seen that

$$\sup_{x \in (A^{-1} \cdot K)^*} \overline{a}'x = 0 \leq \overline{b} < \overline{a}'y, \tag{2.8}$$

implying that $\overline{a} \in \big((A^{-1} \cdot K)^*\big)^*$. Since K is a closed convex cone and A is a linear (and therefore continuous) transformation, the set $A^{-1} \cdot K$ is a closed convex cone. Furthermore, by the Polar Cone Theorem, we have that $\big((A^{-1} \cdot K)^*\big)^* = A^{-1} \cdot K$. Therefore, $\overline{a} \in A^{-1} \cdot K$, implying that $A\overline{a} \in K$. Since $y \in A'K^*$, we have $y = A'v$ for some $v \in K^*$, and it follows that

$$y'\overline{a} = (A'v)'\overline{a} = v'A\overline{a} \leq 0,$$

contradicting Eq. (2.8). Hence, we must have $y \in (A^{-1} \cdot K)^*$, implying that

$$A'K^* \subset (A^{-1} \cdot K)^*.$$

Taking the closure of both sides of this relation, we obtain

$$\mathrm{cl}(A'K^*) \subset (A^{-1} \cdot K)^*,$$

completing the proof.

Suppose that $\mathrm{ri}(K^*) \cap R(A) \neq \emptyset$. We will show that the cone $A'K^*$ is closed by using Prop. 1.4.13. According to this proposition, if $R_{K^*} \cap N(A')$ is a subspace of the lineality space L_{K^*} of K^*, then

$$\mathrm{cl}(A'K^*) = A'K^*.$$

Thus, it suffices to verify that $R_{K^*} \cap N(A')$ is a subspace of L_{K^*}. Indeed, we will show that $R_{K^*} \cap N(A') = L_{K^*} \cap N(A')$.

Let $y \in K^* \cap N(A')$. Because $y \in K^*$, we obtain

$$(-y)'x \geq 0, \qquad \forall\ x \in K. \tag{2.9}$$

For $y \in N(A')$, we have $-y \in N(A')$ and since $N(A') = R(A)^\perp$, it follows that

$$(-y)'z = 0, \qquad \forall\ z \in R(A). \tag{2.10}$$

In view of the relation $\text{ri}(K) \cap R(A) \neq \emptyset$, and Eqs. (2.9) and (2.10), the linear function $(-y)'x$ attains its minimum over the convex set K at a point in the relative interior of K, implying that $(-y)'x = 0$ for all $x \in K$ (cf. Prop. 1.3.4). Hence $(-y) \in K^*$, so that $y \in L_{K^*}$ and because $y \in N(A')$, we see that $y \in L_{K^*} \cap N(A')$. The reverse inclusion follows directly from the relation $L_{K^*} \subset R_{K^*}$, thus completing the proof.

2.10 (Pointed Cones and Bases)

Let C be a closed convex cone in $\Re^n$. We say that C is a *pointed cone* if $C \cap (-C) = \{0\}$. A convex set $D \subset \Re^n$ is said to be a *base* for C if $C = \text{cone}(D)$ and $0 \notin \text{cl}(D)$. Show that the following properties are equivalent:

(a) C is a pointed cone.

(b) $\text{cl}(C^* - C^*) = \Re^n$.

(c) $C^* - C^* = \Re^n$.

(d) C^* has nonempty interior.

(e) There exist a nonzero vector $\hat{x} \in \Re^n$ and a positive scalar δ such that $\hat{x}'x \geq \delta \|x\|$ for all $x \in C$.

(f) C has a bounded base.

Hint: Use Exercise 2.8 to show the implications (a) ⇒ (b) ⇒ (c) ⇒ (d) ⇒ (e) ⇒ (f) ⇒ (a).

Solution: (a) ⇒ (b) Since C is a pointed cone, $C \cap (-C) = \{0\}$, so that

$$\big(C \cap (-C)\big)^* = \Re^n.$$

On the other hand, by Exercise 2.8, it follows that

$$\big(C \cap (-C)\big)^* = \text{cl}(C^* - C^*),$$

which when combined with the preceding relation yields $\text{cl}(C^* - C^*) = \Re^n$.

(b) ⇒ (c) Since C is a closed convex cone, by the polar cone operations of Exercise 2.8, it follows that

$$\big(C \cap (-C)\big)^* = \text{cl}(C^* - C^*) = \Re^n.$$

By taking the polars and using the Polar Cone Theorem (Prop. 2.2.1), we obtain

$$\Big(\big(C \cap (-C)\big)^*\Big)^* = C \cap (-C) = \{0\}. \tag{2.11}$$

Now, to arrive at a contradiction assume that there is a vector $\hat{x} \in \Re^n$ such that $\hat{x} \notin C^* - C^*$. Then, by the Separating Hyperplane Theorem (Prop. 1.5.2), there exists a nonzero vector $a \in \Re^n$ such that

$$a'\hat{x} \geq a'x, \qquad \forall\, x \in C^* - C^*.$$

If $a'x > 0$ for some $x \in C^* - C^*$, then since $C^* - C^*$ is a cone, the right hand-side of the preceding relation can be arbitrarily large, a contradiction. Thus, we have $a'x \leq 0$ for all $x \in C^* - C^*$, implying that $a \in (C^* - C^*)^*$. By the polar cone operations of Exercise 2.8(b) and the Polar Cone Theorem, it follows that

$$(C^* - C^*)^* = (C^*)^* \cap (-C^*)^* = C \cap (-C).$$

Thus, $a \in C \cap (-C)$ with $a \neq 0$, contradicting Eq. (2.11). Hence, we must have $C^* - C^* = \Re^n$.

(c) ⇒ (d) Because $C^* \subset \text{aff}(C^*)$ and $-C^* \subset \text{aff}(C^*)$, we have $C^* - C^* \subset \text{aff}(C^*)$ and since $C^* - C^* = \Re^n$, it follows that $\text{aff}(C^*) = \Re^n$, showing that C^* has nonempty interior.

(d) ⇒ (e) Let v be a vector in the interior of C^*. Then, there exists a positive scalar δ such that the vector $v + \delta \frac{y}{\|y\|}$ is in C^* for all $y \in \Re^n$ with $y \neq 0$, i.e.,

$$\left(v + \delta \frac{y}{\|y\|}\right)' x \leq 0, \qquad \forall\, x \in C, \quad \forall\, y \in \Re^n,\ y \neq 0.$$

By taking $y = x$, it follows that

$$\left(v + \delta \frac{x}{\|x\|}\right)' x \leq 0, \qquad \forall\, x \in C,\ x \neq 0,$$

implying that

$$v'x + \delta\|x\| \leq 0, \qquad \forall\, x \in C,\ x \neq 0.$$

Clearly, this relation holds for $x = 0$, so that

$$v'x \leq -\delta\|x\|, \qquad \forall\, x \in C.$$

Multiplying the preceding relation with -1 and letting $\hat{x} = -v$, we obtain

$$\hat{x}'x \geq \delta\|x\|, \qquad \forall\, x \in C.$$

(e) ⇒ (f) Let

$$D = \big\{y \in C \mid \hat{x}'y = 1\big\}.$$

Then, D is a closed convex set since it is the intersection of the closed convex cone C and the closed convex set $\{y \mid \hat{x}'y = 1\}$. Obviously, $0 \notin D$. Thus, to show that D is a base for C, it remains to prove that $C = \text{cone}(D)$. Take any $x \in C$. If $x = 0$, then $x \in \text{cone}(D)$ and we are done, so assume that $x \neq 0$. We have by hypothesis

$$\hat{x}'x \geq \delta\|x\| > 0, \qquad \forall\, x \in C,\ x \neq 0,$$

so we may define $\hat{y} = \frac{x}{\hat{x}'x}$. Clearly, $\hat{y} \in D$ and $x = (\hat{x}'x)\hat{y}$ with $\hat{x}'x > 0$, showing that $x \in \text{cone}(D)$ and that $C \subset \text{cone}(D)$. Since $D \subset C$, the inclusion $\text{cone}(D) \subset C$ is obvious. Thus, $C = \text{cone}(D)$ and D is a base for C. Furthermore, for every y in D, since y is also in C, we have

$$1 = \hat{x}'y \geq \delta\|y\|,$$

showing that D is bounded and completing the proof.

(f) $\Rightarrow$ (a) Since C has a bounded base, $C = \text{cone}(D)$ for some bounded convex set D with $0 \notin \text{cl}(D)$. To arrive at a contradiction, we assume that the cone C is not pointed, so that there exists a nonzero vector $d \in C \cap (-C)$, implying that d and $-d$ are in C. Let $\{\lambda_k\}$ be a sequence of positive scalars. Since $\lambda_k d \in C$ for all k and D is a base for C, there exist a sequence $\{\mu_k\}$ of positive scalars and a sequence $\{y_k\}$ of vectors in D such that

$$\lambda_k d = \mu_k y_k, \qquad \forall\ k.$$

Therefore, $y_k = \frac{\lambda_k}{\mu_k} d \in D$ for all k and because D is bounded, the sequence $\big\{y_k\big\}$ has a subsequence converging to some $y \in \text{cl}(D)$. Without loss of generality, we may assume that $y_k \to y$, which in view of $y_k = \frac{\lambda_k}{\mu_k} d$ for all k, implies that $y = \alpha d$ and $\alpha d \in \text{cl}(D)$ for some $\alpha \geq 0$. Furthermore, by the definition of base, we have $0 \notin \text{cl}(D)$, so that $\alpha > 0$. Similar to the preceding, by replacing d with $-d$, we can show that $\tilde{\alpha}(-d) \in \text{cl}(D)$ for some positive scalar $\tilde{\alpha}$. Therefore, $\alpha d \in \text{cl}(D)$ and $\tilde{\alpha}(-d) \in \text{cl}(D)$ with $\alpha > 0$ and $\tilde{\alpha} > 0$. Since D is convex, its closure $\text{cl}(D)$ is also convex, implying that $0 \in \text{cl}(D)$, contradicting the definition of a base. Hence, the cone C must be pointed.

2.11

Show that a closed convex cone is polyhedral if and only if its polar cone is polyhedral.

Solution: Let the closed convex cone C be polyhedral, and of the form

$$C = \big\{x \mid a_j'x \leq 0,\ j = 1, \ldots, r\big\},$$

for some vectors a_j in $\Re^n$. By Farkas' Lemma, we have

$$C^* = \text{cone}\big(\{a_1, \ldots, a_r\}\big),$$

so the polar cone of a polyhedral cone is finitely generated. Conversely, using the Polar Cone Theorem, we have

$$\text{cone}\big(\{a_1,\dots,a_r\}\big)^* = \big\{x \mid a_j'x \le 0,\ j=1,\dots,r\big\},$$

so the polar of a finitely generated cone is polyhedral. Thus, a closed convex cone is polyhedral if and only if its polar cone is finitely generated. By the Minkowski-Weyl Theorem (Prop. 2.3.2), a cone is finitely generated if and only if it is polyhedral. Therefore, a closed convex cone is polyhedral if and only if its polar cone is polyhedral.

2.12 (Closedness of Finitely Generated Cones)

This exercise proves that a finitely generated cone is closed without invoking Prop. 1.4.13. Let $a_1,\dots,a_r$ be vectors in $\Re^n$ and let A be the $n\times r$ matrix that has as columns these vectors. Consider the cone generated by $a_1,\dots,a_r$:

$$\text{cone}\big(\{a_1,\dots,a_r\}\big) = \{A\mu \mid \mu \ge 0\}.$$

(a) Show that if $a_1,\dots,a_r$ are linearly independent, then $\text{cone}\big(\{a_1,\dots,a_r\}\big)$ is closed. *Hint*: Show that if $y_k = \{A\mu_k\}$ and $y_k \to y$, then $y = A\mu$ with

$$\mu = \lim_{k\to\infty} \mu_k = \lim_{k\to\infty} (A'A)^{-1}A'y_k = (A'A)^{-1}A'y.$$

(b) Show that $\text{cone}\big(\{a_1,\dots,a_r\}\big)$ is closed without the linear independence assumption of part (a). *Hint*: Use Caratheodory's Theorem to show that $\text{cone}\big(\{a_1,\dots,a_r\}\big)$ is equal to the union of a finite number of cones generated by linearly independent vectors.

Solution: (a) Consider a sequence $\{y_k\} \subset \text{cone}\big(\{a_1,\dots,a_r\}\big)$ with $y_k \to y$. We will show that $y \in \text{cone}\big(\{a_1,\dots,a_r\}\big)$. For each k, we have $y_k = A\mu_k$ for some $\mu_k \ge 0$, from which we obtain,

$$A'y_k = A'A\mu_k.$$

Since $a_1,\dots,a_r$ are assumed linearly independent, the matrix $A'A$ is invertible, and we have

$$\mu_k = (A'A)^{-1}A'y_k.$$

It follows that

$$\mu_k \to \mu,$$

where

$$\mu = (A'A)^{-1}A'y.$$

Furthermore, since $\mu_k \ge 0$, we have $\mu \ge 0$. Taking the limit in the relation $y_k = A\mu_k$, we obtain $y = A\mu$ with $\mu \ge 0$, so $y \in \text{cone}\big(\big\{a_1,\dots,a_r\big\}\big)$.

(b) By Caratheodory's Theorem, every vector in $\text{cone}\big(\{a_1,\dots,a_r\}\big)$ is a positive combination of linearly independent vectors. Thus, $\text{cone}\big(\{a_1,\dots,a_r\}\big)$ is the union of $\text{cone}\big(\{a_j \mid j\in J\}\big)$ as J ranges over all subsets of $\{1,\dots,r\}$ such that the set $\{a_j \mid j \in J\}$ is linearly independent. Each of these cones is closed by part (a), so their union is also closed.

2.13

Let P be a polyhedral set in $\Re^n$, with a Minkowski-Weyl Representation

$$P = \left\{ x \;\middle|\; x = \sum_{j=1}^{m} \mu_j v_j + y,\ \sum_{j=1}^{m} \mu_j = 1,\ \mu_j \geq 0,\ j = 1, \ldots, m,\ y \in C \right\},$$

where $v_1, \ldots, v_m$ are some vectors in $\Re^n$ and C is a finitely generated cone in $\Re^n$ (cf. Prop. 2.3.3). Show that:

(a) The recession cone of P is equal to C.

(b) Each extreme point of P is equal to some vector v_i that cannot be represented as a convex combination of the vectors v_j with $v_j \neq v_i$.

Solution: (a) We first show that C is a subset of R_P, the recession cone of P. Let $\overline{y} \in C$, and choose any $\alpha \geq 0$ and $x \in P$ of the form $x = \sum_{j=1}^{m} \mu_j v_j$. Since C is a cone, $\alpha \overline{y} \in C$, so that $x + \alpha \overline{y} \in P$ for all $\alpha \geq 0$. It follows that $\overline{y} \in R_P$. Hence $C \subset R_P$. Conversely, to show that $R_P \subset C$, let $\overline{y} \in R_P$ and take any $x \in P$. Then $x + k\overline{y} \in P$ for all $k \geq 1$. Since $P = V + C$, where $V = \text{conv}\big(\{v_1, \ldots, v_m\}\big)$, it follows that

$$x + k\overline{y} = v^k + y^k, \qquad \forall\ k \geq 1,$$

with $v^k \in V$ and $y^k \in C$ for all $k \geq 1$. Because V is compact, the sequence $\{v^k\}$ has a limit point $v \in V$, and without loss of generality, we may assume that $v^k \to v$. Then

$$\lim_{k \to \infty} \|k\overline{y} - y^k\| = \lim_{k \to \infty} \|v^k - x\| = \|v - x\|,$$

implying that

$$\lim_{k \to \infty} \big\|\overline{y} - (1/k)y^k\big\| = 0.$$

Therefore, the sequence $\big\{(1/k)y^k\big\}$ converges to $\overline{y}$. Since $y^k \in C$ for all $k \geq 1$, the sequence $\big\{(1/k)y^k\big\}$ is in C, and by the closedness of C, it follows that $\overline{y} \in C$. Hence, $R_P \subset C$.

(b) Any point in P has the form $v + y$ with $v \in \text{conv}\big(\{v_1, \ldots, v_m\}\big)$ and $y \in C$, or equivalently

$$v + y = \frac{1}{2}v + \frac{1}{2}(v + 2y),$$

with v and $v + 2y$ being two distinct points in P if $y \neq 0$. Therefore, none of the points $v + y$, with $v \in \text{conv}\big(\{v_1, \ldots, v_m\}\big)$ and $y \in C$, is an extreme point of P if $y \neq 0$. Hence, an extreme point of P must be in the set $\{v_1, \ldots, v_m\}$. Since by definition, an extreme point of P is not a convex combination of points in P, an extreme point of P must be equal to some v_i that cannot be expressed as a convex combination of the remaining vectors v_j, $j \neq i$.

2.14 (Cones Generated by Polyhedral Sets)

Show that if P is a polyhedral set in $\Re^n$ containing the origin, then cone(P) is a polyhedral cone. Give an example showing that if P does not contain the origin, then cone(P) may not be a polyhedral cone.

Solution: We give two proofs. The first is based on the Minkowski-Weyl Representation of a polyhedral set P (cf. Prop. 2.3.3), while the second is based on a representation of P by a system of linear inequalities.

Let P be a polyhedral set with Minkowski-Weyl representation

$$P = \left\{ x \;\Big|\; x = \sum_{j=1}^{m} \mu_j v_j + y,\ \sum_{j=1}^{m} \mu_j = 1,\ \mu_j \ge 0,\ j = 1, \ldots, m,\ y \in C \right\},$$

where $v_1, \ldots, v_m$ are some vectors in $\Re^n$ and C is a finitely generated cone in $\Re^n$. Let C be given by

$$C = \left\{ y \;\Big|\; y = \sum_{i=1}^{r} \lambda_i a_i,\ \lambda_i \ge 0,\ i = 1, \ldots, r \right\},$$

where $a_1, \ldots, a_r$ are some vectors in $\Re^n$, so that

$$P = \left\{ x \;\Big|\; x = \sum_{j=1}^{m} \mu_j v_j + \sum_{i=1}^{r} \lambda_i a_i,\ \sum_{j=1}^{m} \mu_j = 1,\ \mu_j \ge 0,\ \forall\ j,\ \lambda_i \ge 0,\ \forall\ i \right\}.$$

We claim that

$$\text{cone}(P) = \text{cone}\big(\{v_1, \ldots, v_m, a_1, \ldots, a_r\}\big).$$

Since $P \subset \text{cone}\big(\{v_1, \ldots, v_m, a_1, \ldots, a_r\}\big)$, it follows that

$$\text{cone}(P) \subset \text{cone}\big(\{v_1, \ldots, v_m, a_1, \ldots, a_r\}\big).$$

Conversely, let $y \in \text{cone}\big(\{v_1, \ldots, v_m, a_1, \ldots, a_r\}\big)$. Then, we have

$$y = \sum_{j=1}^{m} \overline{\mu}_j v_j + \sum_{i=1}^{r} \overline{\lambda}_i a_i,$$

with $\overline{\mu}_j \ge 0$ and $\overline{\lambda}_i \ge 0$ for all i and j. If $\overline{\mu}_j = 0$ for all j, then $y = \sum_{i=1}^{r} \overline{\lambda}_i a_i \in C$, and since $C = R_P$ (cf. Exercise 2.13), it follows that $y \in R_P$. Because the origin belongs to P and $y \in R_P$, we have $0 + y \in P$, implying that $y \in P$, and consequently $y \in \text{cone}(P)$. If $\overline{\mu}_j > 0$ for some j, then by setting $\overline{\mu} = \sum_{j=1}^{m} \overline{\mu}_j$, $\mu_j = \overline{\mu}_j / \overline{\mu}$ for all j, and $\lambda_i = \overline{\lambda}_i / \overline{\mu}$ for all i, we obtain

$$y = \overline{\mu} \left(\sum_{j=1}^{m} \mu_j v_j + \sum_{i=1}^{r} \lambda_i a_i \right),$$

where $\overline{\mu} > 0$, $\mu_j \geq 0$ with $\sum_{j=1}^m \mu_j = 1$, and $\lambda_i \geq 0$. Therefore $y = \overline{\mu}\, \overline{x}$ with $\overline{x} \in P$ and $\overline{\mu} > 0$, implying that $y \in \text{cone}(P)$ and showing that

$$\text{cone}\Big(\{v_1, \ldots, v_m, a_1, \ldots, a_r\}\Big) \subset \text{cone}(P).$$

We now give an alternative proof using the representation of P by a system of linear inequalities. Let P be given by

$$P = \Big\{x \mid a_j'x \leq b_j,\ j = 1, \ldots, r\Big\},$$

where $a_1, \ldots, a_r$ are vectors in $\Re^n$ and $b_1, \ldots, b_r$ are scalars. Since P contains the origin, it follows that $b_j \geq 0$ for all j. Define the index set J as follows

$$J = \{j \mid b_j = 0\}.$$

We consider separately the two cases where $J \neq \emptyset$ and $J = \emptyset$. If $J \neq \emptyset$, then we will show that

$$\text{cone}(P) = \Big\{x \mid a_j'x \leq 0,\ j \in J\Big\}.$$

To see this, note that since $P \subset \Big\{x \mid a_j'x \leq 0,\ j \in J\Big\}$, we have

$$\text{cone}(P) \subset \Big\{x \mid a_j'x \leq 0,\ j \in J\Big\}.$$

Conversely, let $\overline{x} \in \Big\{x \mid a_j'x \leq 0,\ j \in J\Big\}$. We will show that $\overline{x} \in \text{cone}(P)$. If $\overline{x} \in P$, then $\overline{x} \in \text{cone}(P)$ and we are done, so assume that $\overline{x} \notin P$, implying that the set

$$\overline{J} = \{j \notin J \mid a_j'\overline{x} > b_j\} \tag{2.12}$$

is nonempty. By the definition of J, we have $b_j > 0$ for all $j \notin J$, so let

$$\mu = \min_{j \in \overline{J}} \frac{b_j}{a_j'\overline{x}},$$

and note that $0 < \mu < 1$. We have

$$a_j'(\mu\overline{x}) \leq 0, \qquad \forall\ j \in J,$$
$$a_j'(\mu\overline{x}) \leq b_j, \qquad \forall\ j \in \overline{J}.$$

For $j \notin \overline{J} \cup J$ and $a_j'\overline{x} \leq 0 < b_j$, since $\mu > 0$, we still have $a_j'(\mu\overline{x}) \leq 0 < b_j$. For $j \notin \overline{J} \cup J$ and $0 < a_j'\overline{x} \leq b_j$, since $\mu < 1$, we have $0 < a_j'(\mu\overline{x}) < b_j$. Therefore, $\mu\overline{x} \in P$, implying that $\overline{x} = \frac{1}{\mu}(\mu\overline{x}) \in \text{cone}(P)$. It follows that

$$\Big\{x \mid a_j'x \leq 0,\ j \in J\Big\} \subset \text{cone}(P),$$

and hence, $\text{cone}(P) = \Big\{x \mid a_j'x \leq 0,\ j \in J\Big\}$.

If $J = \emptyset$, then we will show that $\text{cone}(P) = \Re^n$. To see this, take any $\overline{x} \in \Re^n$. If $\overline{x} \in P$, then clearly $\overline{x} \in \text{cone}(P)$, so assume that $\overline{x} \notin P$, implying that the set $\overline{J}$ as defined in Eq. (2.12) is nonempty. Note that $b_j > 0$ for all j, since J is empty. The rest of the proof is similar to the preceding case.

As an example, where cone(P) is not polyhedral when P does not contain the origin, consider the polyhedral set $P \subset \Re^2$ given by

$$P = \Big\{(x_1, x_2) \mid x_1 \geq 0,\ x_2 = 1\Big\}.$$

Then, we have

$$\text{cone}(P) = \Big\{(x_1, x_2) \mid x_1 > 0,\ x_2 > 0\Big\} \cup \Big\{(x_1, x_2) \mid x_1 = 0, x_2 \geq 0\Big\},$$

which is not closed and therefore not polyhedral.

2.15

Let P be a polyhedral set in $\Re^n$, with a Minkowski-Weyl Representation

$$P = \left\{ x \;\middle|\; x = \sum_{j=1}^{m} \mu_j v_j + y,\ \sum_{j=1}^{m} \mu_j = 1,\ \mu_j \geq 0,\ j = 1, \ldots, m,\ y \in C \right\},$$

where $v_1, \ldots, v_m$ are some vectors in $\Re^n$ and C is a finitely generated cone in $\Re^n$ (cf. Prop. 2.3.3). Show that:

(a) The recession cone of P is equal to C.

(b) Each extreme point of P is equal to some vector v_i that cannot be represented as a convex combination of the remaining vectors v_j, $j \neq i$.

Solution: (a) We first show that C is a subset of R_P, the recession cone of P. Let $\overline{y} \in C$, and choose any $\alpha \geq 0$ and $x \in P$ of the form $x = \sum_{j=1}^{m} \mu_j v_j$. Since C is a cone, $\alpha\overline{y} \in C$, so that $x + \alpha\overline{y} \in P$ for all $\alpha \geq 0$. It follows that $\overline{y} \in R_P$. Hence $C \subset R_P$.

Conversely, to show that $R_P \subset C$, let $\overline{y} \in R_P$ and take any $x \in P$. Then $x + k\overline{y} \in P$ for all $k \geq 1$. Since $P = V + C$, where $V = \text{conv}\big(\{v_1, \ldots, v_m\}\big)$, it follows that

$$x + k\overline{y} = v^k + y^k, \qquad \forall\ k \geq 1,$$

with $v^k \in V$ and $y^k \in C$ for all $k \geq 1$. Because V is compact, the sequence $\{v^k\}$ has a limit point $v \in V$, and without loss of generality, we may assume that $v^k \to v$. Then

$$\lim_{k \to \infty} \|k\overline{y} - y^k\| = \lim_{k \to \infty} \|v^k - x\| = \|v - x\|,$$

implying that

$$\lim_{k \to \infty} \big\|\overline{y} - (1/k)y^k\big\| = 0.$$

Therefore, the sequence $\big\{(1/k)y^k\big\}$ converges to $\overline{y}$. Since $y^k \in C$ for all $k \geq 1$, the sequence $\big\{(1/k)y^k\big\}$ is in C, and by the closedness of C, it follows that $\overline{y} \in C$. Hence, $R_P \subset C$.

(b) Any point in P has the form $v + y$ with $v \in \text{conv}\big(\{v_1, \ldots, v_m\}\big)$ and $y \in C$, or equivalently

$$v + y = \frac{1}{2}v + \frac{1}{2}(v + 2y),$$

with v and $v + 2y$ being two distinct points in P if $y \neq 0$. Therefore, none of the points $v + y$, with $v \in \text{conv}\big(\{v_1, \ldots, v_m\}\big)$ and $y \in C$, is an extreme point of P if $y \neq 0$. Hence, an extreme point of P must be in the set $\{v_1, \ldots, v_m\}$. Since by definition, an extreme point of P is not a convex combination of points in P, an extreme point of P must be equal to some v_i that cannot be expressed as a convex combination of the remaining vectors v_j, $j \neq i$.

2.16

Show that if P is a polyhedral set in $\Re^n$ containing the origin, then cone(P) is a polyhedral cone. Give an example showing that if P does not contain the origin, then cone(P) may not be a polyhedral cone.

Solution: We give two proofs. The first is based on the Minkowski-Weyl Representation of a polyhedral set P (cf. Prop. 2.3.3), while the second is based on a representation of P by a system of linear inequalities.

Let P be a polyhedral set with Minkowski-Weyl representation

$$P = \left\{ x \;\middle|\; x = \sum_{j=1}^{m} \mu_j v_j + y,\ \sum_{j=1}^{m} \mu_j = 1,\ \mu_j \geq 0,\ j = 1, \ldots, m,\ y \in C \right\},$$

where $v_1, \ldots, v_m$ are some vectors in $\Re^n$ and C is a finitely generated cone in $\Re^n$. Let C be given by

$$C = \left\{ y \;\middle|\; y = \sum_{i=1}^{r} \lambda_i a_i,\ \lambda_i \geq 0,\ i = 1, \ldots, r \right\},$$

where $a_1, \ldots, a_r$ are some vectors in $\Re^n$, so that

$$P = \left\{ x \;\middle|\; x = \sum_{j=1}^{m} \mu_j v_j + \sum_{i=1}^{r} \lambda_i a_i,\ \sum_{j=1}^{m} \mu_j = 1,\ \mu_j \geq 0,\ \forall\ j,\ \lambda_i \geq 0,\ \forall\ i \right\}.$$

We claim that

$$\text{cone}(P) = \text{cone}\big(\{v_1, \ldots, v_m, a_1, \ldots, a_r\}\big).$$

Since $P \subset \text{cone}\big(\{v_1, \ldots, v_m, a_1, \ldots, a_r\}\big)$, it follows that

$$\text{cone}(P) \subset \text{cone}\big(\{v_1, \ldots, v_m, a_1, \ldots, a_r\}\big).$$

Conversely, let $y \in \text{cone}\big(\{v_1, \ldots, v_m, a_1, \ldots, a_r\}\big)$. Then, we have

$$y = \sum_{j=1}^{m} \overline{\mu}_j v_j + \sum_{i=1}^{r} \overline{\lambda}_i a_i,$$

with $\overline{\mu}_j \geq 0$ and $\overline{\lambda}_i \geq 0$ for all i and j. If $\overline{\mu}_j = 0$ for all j, then $y = \sum_{i=1}^{r} \overline{\lambda}_i a_i \in C$, and since $C = R_P$ (cf. Exercise 2.15), it follows that $y \in R_P$. Because the origin belongs to P and $y \in R_P$, we have $0 + y \in P$, implying that $y \in P$, and consequently $y \in \text{cone}(P)$. If $\overline{\mu}_j > 0$ for some j, then by setting $\overline{\mu} = \sum_{j=1}^{m} \overline{\mu}_j$, $\mu_j = \overline{\mu}_j / \overline{\mu}$ for all j, and $\lambda_i = \overline{\lambda}_i / \overline{\mu}$ for all i, we obtain

$$y = \overline{\mu} \left(\sum_{j=1}^{m} \mu_j v_j + \sum_{i=1}^{r} \lambda_i a_i \right),$$

where $\overline{\mu} > 0$, $\mu_j \geq 0$ with $\sum_{j=1}^{m} \mu_j = 1$, and $\lambda_i \geq 0$. Therefore $y = \overline{\mu}\ \overline{x}$ with $\overline{x} \in P$ and $\overline{\mu} > 0$, implying that $y \in \text{cone}(P)$ and showing that

$$\text{cone}\Big(\{v_1, \ldots, v_m, a_1, \ldots, a_r\}\Big) \subset \text{cone}(P).$$

We now give an alternative proof using the representation of P by a system of linear inequalities. Let P be given by

$$P = \Big\{x \mid a_j' x \leq b_j,\ j = 1, \ldots, r\Big\},$$

where $a_1, \ldots, a_r$ are vectors in $\Re^n$ and $b_1, \ldots, b_r$ are scalars. Since P contains the origin, it follows that $b_j \geq 0$ for all j. Define the index set J as follows

$$J = \{j \mid b_j = 0\}.$$

We consider separately the two cases where $J \neq \emptyset$ and $J = \emptyset$. If $J \neq \emptyset$, then we will show that

$$\text{cone}(P) = \Big\{x \mid a_j' x \leq 0,\ j \in J\Big\}.$$

To see this, note that since $P \subset \Big\{x \mid a_j' x \leq 0,\ j \in J\Big\}$, we have

$$\text{cone}(P) \subset \Big\{x \mid a_j' x \leq 0,\ j \in J\Big\}.$$

Conversely, let $\overline{x} \in \Big\{x \mid a_j' x \leq 0,\ j \in J\Big\}$. We will show that $\overline{x} \in \text{cone}(P)$. If $\overline{x} \in P$, then $\overline{x} \in \text{cone}(P)$ and we are done, so assume that $\overline{x} \notin P$, implying that the set

$$\overline{J} = \{j \notin J \mid a_j' \overline{x} > b_j\} \tag{2.13}$$

is nonempty. By the definition of J, we have $b_j > 0$ for all $j \notin J$, so let

$$\mu = \min_{j \in \overline{J}} \frac{b_j}{a_j' \overline{x}},$$

and note that $0 < \mu < 1$. We have

$$\begin{aligned} a_j'(\mu \overline{x}) &\leq 0, \qquad \forall\ j \in J, \\ a_j'(\mu \overline{x}) &\leq b_j, \qquad \forall\ j \in \overline{J}. \end{aligned}$$

For $j \notin \overline{J} \cup J$ and $a_j' \overline{x} \leq 0 < b_j$, since $\mu > 0$, we still have $a_j'(\mu \overline{x}) \leq 0 < b_j$. For $j \notin \overline{J} \cup J$ and $0 < a_j' \overline{x} \leq b_j$, since $\mu < 1$, we have $0 < a_j'(\mu \overline{x}) < b_j$. Therefore, $\mu \overline{x} \in P$, implying that $\overline{x} = \frac{1}{\mu}(\mu \overline{x}) \in \text{cone}(P)$. It follows that

$$\Big\{x \mid a_j' x \leq 0,\ j \in J\Big\} \subset \text{cone}(P),$$

and hence, $\text{cone}(P) = \Big\{x \mid a_j' x \leq 0,\ j \in J\Big\}$.

If $J = \emptyset$, then we will show that $\text{cone}(P) = \Re^n$. To see this, take any $\overline{x} \in \Re^n$. If $\overline{x} \in P$, then clearly $\overline{x} \in \text{cone}(P)$, so assume that $\overline{x} \notin P$, implying that the set $\overline{J}$ as defined in Eq. (2.13) is nonempty. Note that $b_j > 0$ for all j, since J is empty. The rest of the proof is similar to the preceding case.

As an example, where cone(P) is not polyhedral when P does not contain the origin, consider the polyhedral set $P \subset \Re^2$ given by

$$P = \Big\{(x_1, x_2) \mid x_1 \geq 0,\ x_2 = 1\Big\}.$$

Then, we have

$$\text{cone}(P) = \Big\{(x_1, x_2) \mid x_1 > 0,\ x_2 > 0\Big\} \cup \Big\{(x_1, x_2) \mid x_1 = 0, x_2 \geq 0\Big\},$$

which is not closed and therefore not polyhedral.

2.17 (Polar Sets)

This exercise introduces a notion of polar set that generalizes the notion of polar cone. Polar sets originated in Euclidean geometry, where they can be used to provide elegant proofs to many classical theorems. Given a nonempty set $C \subset \Re^n$, the *polar set of* C is defined as

$$C^\circ = \{y \mid y'x \leq 1, \forall\, x \in C\}.$$

Thus the polar set C° is the level set $\big\{y \mid \sigma_C(y) \leq 1\big\}$ of the support function σ_C of C. Since a single level set is sufficient to characterize all level sets of a support function (in view of positive homogeneity), it follows from the Conjugacy Theorem (Prop. 1.6.1), that any set is fully characterized by its polar up to convex closure, i.e., two sets with the same polar set have the same convex closure.

(a) Show that C° is a closed convex set. Furthermore, C° is bounded if and only if the origin in an interior point of $\mathrm{conv}(C)$.

(b) Show that the polar set of a cone is equal to its polar cone.

(c) Consider the subset $\hat{C}$ of $\Re^{n+1}$ obtained from C via the lifting procedure,

$$\hat{C} = \big\{(x, 1) \mid x \in C\big\}.$$

Show that C° is obtained from the polar of the cone generated by $\hat{C}$, by "slicing" at the level -1:

$$C^\circ = \Big\{y \mid (y, -1) \in \big(\mathrm{cone}(\hat{C})\big)^*\Big\}.$$

(d) Show that if C is a finite set, then C° is a polyhedral set.

(e) Show that

$$(C^\circ)^\circ = \mathrm{cl}\Big(\mathrm{conv}\big(\{0\} \cup C\big)\Big),$$

so if C is a closed convex set containing the origin, we have $(C^\circ)^\circ = C$.

(f) Consider a bounded polyhedral set P. For each extreme point v of P, consider the halfspace $H_v = \{y \mid y'v \leq 1\}$. Show that the polar set P° is the intersection of the halfspaces H_v, where v ranges over the extreme points of P.

(g) Consider a circle in the plane that is centered at the origin, and a convex polygon that is inscribed in the circle and contains the origin in its interior. Show that the polar set is a polygon that can be circumscribed around some circle centered at the origin.

Solution:
(a) Clearly, we have

$$0 \in \mathrm{int}(\mathrm{conv}(C)) \qquad \Longleftrightarrow \qquad \sigma_C(y) > 0, \qquad \forall\, y \neq 0.$$

Since σ_C is positively homogeneous, it is equal to its recession function, so

$$0 \in \mathrm{int}(\mathrm{conv}(C)) \qquad \Longleftrightarrow \qquad R_{\sigma_C} = \{0\}.$$

Since $R_{\sigma_C} = \{0\}$ if and only if the nonempty level sets of σ_C are compact, and C° is a level set, we have

$$0 \in \mathrm{int}(\mathrm{conv}(C)) \qquad \iff \qquad \big\{y \mid \sigma_C(y) \leq 1\big\} = C^\circ \text{ is compact.}$$

(b) If C is a cone, by Example 5.2.2, σ_C is the indicator function of the polar set C^*. Since C° is a nonempty level set of σ_C, it follows that $C^\circ = C^\star$.

(c) Using the definition of cone:

$$\mathrm{cone}(\hat{C}) = \big\{(\lambda x, \lambda) \mid x \in C,\ \lambda > 0\big\}.$$

Using the definition of polar cone:

$$(\mathrm{cone}(\hat{C}))^\star = \big\{(y, w) \mid y'\lambda x + w'\lambda \leq 0, x \in C, \lambda > 0\big\}$$

Therefore

$$\begin{aligned}\{y \mid (y, -1) \in (\mathrm{cone}(\hat{C}))^\star\} &= \{y \mid y'\lambda x - \lambda \leq 0, x \in C, \lambda > 0\}\\ &= \{y \mid y'x \leq 1\}\\ &= C^\circ\end{aligned}$$

(d) If C is a finite set, C° is the intersection of a finite number of halfspaces. Furthermore, C° is nonempty since it contains the origin, so it is polyhedral.

(e) We first show that

$$C^\circ = (\mathrm{cl}(C)^\circ) = (\mathrm{conv}(C)^\circ) = (\{0\} \cap C)^\circ.$$

The first two equations hold because C, $\mathrm{cl}(C)$, and $\mathrm{conv}(C)$ have the same support function. The third equation is true by the definition of polar set.

Assume that C is closed, convex, and contains the origin. To show that $(C^\circ)^\circ = C$, note that

$$\begin{aligned}(C^\circ)^\circ &= \{x \mid x'y \leq 1,\ \forall\ y \in C^\circ\}\\ &= \big\{x \mid (x, 1)'(y, -1) \leq 0,\ \forall\ (y, -1) \in D^*\big\},\end{aligned}$$

where $D = \mathrm{cone}(\hat{C})$ and the second equation follows from part (c). Since D is a cone, its polar set is a cone by part (b). We write the above equation as

$$(C^\circ)^\circ = \big\{x \mid (x, 1)'(\lambda y, -\lambda) \leq 0,\ \forall\ (y, -1) \in D^*,\ \forall\ \lambda > 0)\big\},$$

or equivalently,

$$(C^\circ)^\circ = \big\{x \mid (x, 1)'(\bar{y}) \leq 0,\ \forall\ \bar{y} \in D^\star\big\},$$

and note that

$$\big\{(\lambda x, \lambda) \mid (x, 1)'\bar{y} \leq 0,\ \forall\ \bar{y} \in D^\star\big\} = (D^*)^*$$

and $(D^*)^* = D$ because $\hat{C}$ is closed and convex. Now it follows that

$$(C^\circ)^\circ = \big\{x \mid (x,1) \in (D^\star)^\star = D\big\}$$

where D is the cone of "lifted" C. Therefore $(C^\circ)^\circ = C$. For an arbitrary set C without any assumption,

$$\mathrm{cl}(\mathrm{conv}(\{0\} \cap C)) = \big(\mathrm{cl}(\mathrm{conv}(\{0\} \cap C)^\circ)\big)^\circ = (C^\circ)^\circ.$$

(f) By the Minkowski-Weyl representation, a bounded polyhedral set P is the convex hull of its extreme points. Thus

$$P\circ = \{y \mid y'x \le 1,\ \forall\, x \in \mathrm{conv}(\{v_1, v_2, ..., v_r\})\}.$$

We have

$$y'x \le 1, \qquad \forall\, x \in \mathrm{conv}(\{v_1, ..., v_r\}) \qquad \Longleftrightarrow \qquad y'v_i \le 1,\ \forall\, i = 1, ..., r.$$

Therefore

$$P\circ = \{y \mid y'x \le 1,\ \forall\, x \in \mathrm{conv}(\{v_1, v_2, ..., v_r\})\} = H_{v_1} \cap ... \cap H_{v_r}.$$

(g) Using part (f), the polar set of the convex polygon P is the intersection of $H_v = \{y \mid y'v \le 1\}$, where v ranges over the extreme points of P. Furthermore, P° is bounded because 0 is an interior point of P, and it is polyhedral because it is the intersection of a finite number of halfspaces.

If P is inscribed in the circle $\{x \mid \|x\| = r\}$, all extreme points v satisfy $\|v\| = r$, and H_v corresponds to a tangent hyperplane of the circle centered at origin with radius $1/r$. Thus, the intersection of H_v can be circumscribed around the circle $\{x \mid \|x\| = 1/r\}$.

2.18 (Support Function of a Polyhedral Set)

Show that the support function of a polyhedral set is a polyhedral function.

Solution: Let X be a polyhedral set with Minkowski-Weyl representation

$$X = \mathrm{conv}\big(\{v_1, \ldots, v_m\}\big) + \mathrm{cone}\big(\{d_1, \ldots, d_r\}\big)$$

for some vectors $v_1, \ldots, v_m, d_1, \ldots, d_r$ (cf. Prop. 2.3.3). The support function of X takes the form

$$\begin{aligned}
\sigma_X(y) &= \sup_{x \in X} y'x \\
&= \sup_{\substack{\alpha_1,\ldots,\alpha_m,\beta_1,\ldots,\beta_r \ge 0 \\ \sum_{i=1}^m \alpha_i = 1}} \left\{ \sum_{i=1}^m \alpha_i v_i' y + \sum_{j=1}^r \beta_j d_j' y \right\} \\
&= \begin{cases} \max_{i=1,\ldots,m} v_i' y & \text{if } d_j' y \le 0,\ j = 1, \ldots, r, \\ \infty & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus the support function is polyhedral.

2.19 (Conjugate of a Polyhedral Function)

Show that that the conjugate of a polyhedral function is polyhedral.

Solution: We first show how the conjugate of a function can be specified in terms of the support function of its epigraph. To derive the corresponding formula, note that the expression for the conjugate of f,

$$h(y) = \sup_{x \in \Re^n} \big\{ x'y - f(x) \big\},$$

can equivalently be written as

$$h(y) = \sup_{(x,w) \in \mathrm{epi}(f)} \{ x'y - w \}.$$

Since the expression in braces in the right-hand side is the inner product of the vectors (x, w) and $(y, -1)$, the supremum above is the value of the support function of epi(f) at $(y, -1)$:

$$h(y) = \sigma_{\mathrm{epi}(f)}(y, -1), \qquad \forall\, y \in \Re^n.$$

Let us apply the preceding result to the case where f is a polyhedral function, so that epi(f) is a polyhedral set. From Exercise 2.18, the support function $\sigma_{\mathrm{epi}(f)}$ is a polyhedral function, and it can be seen that $\sigma_{\mathrm{epi}(f)}(y, -1)$, viewed as a function of y, is polyhedral.