

*Convex Optimization Theory*

*Chapter 4*

*Exercises and Solutions*

Dimitri P. Bertsekas

Massachusetts Institute of Technology

Athena Scientific, Belmont, Massachusetts

<http://www.athenasc.com>

CHAPTER 4: EXERCISES AND SOLUTIONS†

4.1

Let  $F : \mathfrak{R}^{n+m} \mapsto (-\infty, \infty]$  be a proper function, and let  $p$  be the function defined by

$$p(u) = \inf_{x \in \mathfrak{R}^n} F(x, u), \quad u \in \mathfrak{R}^m.$$

Consider the alternative min common/max crossing framework where

$$M = \{(u, w) \mid \text{there is an } x \text{ such that } F(x, u) \leq w\}.$$

Show that the optimal values of the corresponding min common and max crossing problems are the same as those corresponding to  $M = \text{epi}(p)$ .

**Solution:** Let

$$M = \{(u, w) \mid \text{there is an } x \text{ such that } F(x, u) \leq w\}.$$

Then the corresponding min common value is

$$\inf_{\{(x,u) \mid F(x,u) \leq w\}} w = \inf_x F(x, 0) = p(0).$$

Since  $p(0)$  is the min common value corresponding to  $\text{epi}(p)$ , the min common values corresponding to the two choices for  $M$  are equal. Similarly, we show that the cost functions of the max crossing problem corresponding to the two choices for  $M$  are equal.

4.2

In the context of Exercise 4.1, let  $F(x, u) = f_1(x) - f_2(Ax + u)$ , where  $A$  is an  $m \times n$  matrix, and  $f_1 : \mathfrak{R}^n \mapsto (-\infty, \infty]$  and  $f_2 : \mathfrak{R}^m \mapsto (-\infty, \infty]$  are closed convex functions. Show that the dual optimal value is

$$q^* = -f_1^*(A'\lambda) - f_2^*(-\lambda),$$

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† Many of the exercises and solutions given here were developed as part of my earlier convex optimization book [BNO03] (coauthored with Angelia Nedić and Asuman Ozdaglar), and are posted on the internet of that book's web site. The contribution of my coauthors in the development of these exercises and their solutions is gratefully acknowledged. Since some of the exercises and/or their solutions have been modified and also new exercises have been added, all errors are my sole responsibility.

where  $f_1^*$  and  $f_2^*$  are the conjugate functions of  $f_1$  and  $f_2$ , respectively. *Note:* This is the Fenchel duality framework discussed in Section 5.3.5.

**Solution:** In the notation of Exercise 4.1, the max crossing value is

$$q^* = \sup_{\lambda} \{-p^*(-\lambda)\}.$$

By using the change of variables  $z = Ax + u$  in the following calculation, we have

$$\begin{aligned} p^*(-\lambda) &= -\sup_u \{-\lambda'u - \inf_x \{f_1(x) - f_2(Ax + u)\}\} \\ &= \sup_{z,x} \{-\lambda'(z - Ax) - f_1(x) + f_2(z)\} \\ &= f_1^*(A'\lambda) + f_2^*(-\lambda), \end{aligned}$$

where  $f_1^*$  and  $f_2^*$  are the conjugate functions of  $f_1$  and  $f_2$ , respectively. Thus,

$$q^* = -f_1^*(A'\lambda) - f_2^*(-\lambda).$$

### 4.3 (An Example of Lagrangian Duality)

Consider the problem

$$\begin{aligned} &\text{minimize } f(x) \\ &\text{subject to } x \in X, \quad e_i'x = d_i, \quad i = 1, \dots, m, \end{aligned} \tag{4.1}$$

where  $f : \mathfrak{R}^n \mapsto \mathfrak{R}$  is a convex function,  $X$  is a nonempty convex set, and  $e_i$  and  $d_i$  are given vectors and scalars, respectively. Consider the min common/max crossing framework where  $M$  is the subset of  $\mathfrak{R}^{m+1}$  given by

$$M = \left\{ (e_1'x - d_1, \dots, e_m'x - d_m, f(x)) \mid x \in X \right\},$$

and assume that  $w^* < \infty$ .

- (a) Show that  $w^*$  is equal to the optimal value of problem (4.1), and that the max crossing problem is to maximize  $q(\mu)$  given by

$$q(\mu) = \inf_{x \in X} \left\{ f(x) + \sum_{i=1}^m \mu_i (e_i'x - d_i) \right\}.$$

- (b) Show that the corresponding set  $\overline{M}$  is convex.  
(c) Show that if  $X$  is compact, then  $q^* = w^*$ .  
(d) Show that if there exists a vector  $\bar{x} \in \text{ri}(X)$  such that  $e_i'\bar{x} = d_i$  for all  $i = 1, \dots, m$ , then  $q^* = w^*$  and the max crossing problem has an optimal solution.

**Solution:** (a) It is easily seen that  $w^*$  is the minimal value of  $f(x)$  subject to  $x \in \text{conv}(X)$  and  $e'_i x = d_i$ ,  $i = 1, \dots, m$ . The corresponding max crossing problem is given by

$$q^* = \sup_{\mu \in \mathfrak{R}^m} q(\mu),$$

where  $q(\mu)$  is given by

$$q(\mu) = \inf_{(u,w) \in M} \{w + \mu'u\} = \inf_{x \in X} \left\{ f(x) + \sum_{i=1}^m \mu_i (e'_i x - d_i) \right\}.$$

(b) Consider the set

$$\overline{M} = \left\{ (u_1, \dots, u_m, w) \mid \text{there exists } x \in X \text{ such that } e'_i x - d_i = u_i, \forall i, f(x) \leq w \right\}.$$

We show that  $\overline{M}$  is convex. To this end, we consider vectors  $(u, w) \in \overline{M}$  and  $(\tilde{u}, \tilde{w}) \in \overline{M}$ , and we show that their convex combinations lie in  $\overline{M}$ . The definition of  $\overline{M}$  implies that for some  $x \in X$  and  $\tilde{x} \in X$ , we have

$$\begin{aligned} f(x) &\leq w, & e'_i x - d_i &= u_i, & i &= 1, \dots, m, \\ f(\tilde{x}) &\leq \tilde{w}, & e'_i \tilde{x} - d_i &= \tilde{u}_i, & i &= 1, \dots, m. \end{aligned}$$

For any  $\alpha \in [0, 1]$ , we multiply these relations with  $\alpha$  and  $1-\alpha$ , respectively, and add. By using the convexity of  $f$ , we obtain

$$\begin{aligned} f(\alpha x + (1-\alpha)\tilde{x}) &\leq \alpha f(x) + (1-\alpha)f(\tilde{x}) \leq \alpha w + (1-\alpha)\tilde{w}, \\ e'_i(\alpha x + (1-\alpha)\tilde{x}) - d_i &= \alpha u_i + (1-\alpha)\tilde{u}_i, & i &= 1, \dots, m. \end{aligned}$$

In view of the convexity of  $X$ , we have  $\alpha x + (1-\alpha)\tilde{x} \in X$ , so these equations imply that the convex combination of  $(u, w)$  and  $(\tilde{u}, \tilde{w})$  belongs to  $\overline{M}$ , thus proving that  $\overline{M}$  is convex.

(c) We prove this result by showing that all the assumptions of Min Common/Max Crossing Theorem I are satisfied. By assumption,  $w^*$  is finite. It follows from part (b) that the set  $\overline{M}$  is convex. Therefore, we only need to show that for every sequence  $\{(u_k, w_k)\} \subset M$  with  $u_k \rightarrow 0$ , there holds  $w^* \leq \liminf_{k \rightarrow \infty} w_k$ . Consider a sequence  $\{(u_k, w_k)\} \subset M$  with  $u_k \rightarrow 0$ . Since  $X$  is compact and  $f$  is convex by assumption (which implies that  $f$  is continuous by Prop. 1.4.6), it follows from Prop. 1.1.9(c) that set  $M$  is compact. Hence, the sequence  $\{(u_k, w_k)\}$  has a subsequence that converges to some  $(0, \bar{w}) \in M$ . Assume without loss of generality that  $\{(u_k, w_k)\}$  converges to  $(0, \bar{w})$ . Since  $(0, \bar{w}) \in M$ , we get

$$w^* = \inf_{(0,w) \in M} w \leq \bar{w} = \liminf_{k \rightarrow \infty} w_k,$$

proving the desired result, and thus showing that  $q^* = w^*$ .

(d) We prove this result by showing that all the assumptions of Min Common/Max Crossing Theorem II are satisfied. By assumption,  $w^*$  is finite. It

follows from part (b) that the set  $\overline{M}$  is convex. Therefore, we only need to show that the set

$$D = \{(e'_1x - d_1, \dots, e'_mx - d_m) \mid x \in X\}$$

contains the origin in its relative interior. The set  $D$  can equivalently be written as

$$D = E \cdot X - d,$$

where  $E$  is a matrix, whose rows are the vectors  $e'_i$ ,  $i = 1, \dots, m$ , and  $d$  is a vector with entries equal to  $d_i$ ,  $i = 1, \dots, m$ . By Prop. 1.3.6, it follows that

$$\text{ri}(D) = E \cdot \text{ri}(X) - d.$$

Hence the assumption that there exists a vector  $\bar{x} \in \text{ri}(X)$  such that  $E\bar{x} - d = 0$  implies that  $0 \in \text{ri}(D)$ , thus showing that  $q^* = w^*$  and that the max crossing problem has an optimal solution.

#### 4.4 (Lagrangian Duality and Compactness of the Constraint Set)

Consider the problem of Exercise 4.3, but assume that  $f$  is linear and  $X$  is compact (instead of  $f$  and  $X$  being convex). Show that  $q^*$  is equal to the minimal value of  $f(x)$  subject to  $x \in \text{conv}(X)$  and  $e'_ix = d_i$ ,  $i = 1, \dots, m$ . *Hint:* Show that

$$\text{conv}(M) = \left\{ (e'_1x - d_1, \dots, e'_mx - d_m, f(x)) \mid x \in \text{conv}(X) \right\},$$

and use Exercise 4.3(c).

**Solution:** Clearly, the max crossing values corresponding to  $M$  and  $\text{conv}(M)$  are equal [this is true generically, since closed halfspaces containing  $M$  also contain  $\text{conv}(M)$ ]. The expression for  $\text{conv}(M)$  in the hint follows from the linearity of  $f$ . Thus, the min common/max crossing framework for  $\text{conv}(M)$  corresponds to the problem of minimizing  $f(x)$  subject to  $x \in \text{conv}(X)$  and  $e'_ix = d_i$ ,  $i = 1, \dots, m$ . Since  $M$  is compact,  $\text{conv}(M)$  is also compact, and the result follows from Exercise 4.3(c).